UNIT-1

classmate
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VECTOR CALCULUS

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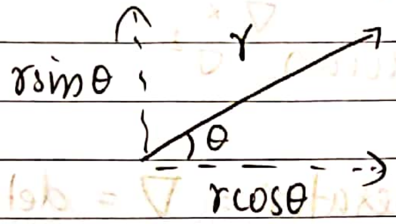
Book: BV Ramana

Vector
- carrier
- variable quantity that can be resolved into components.



- application by honeybees: dance (nectar)
- vector images: can be zoomed without changing image quality and image size

Vector Algebra

1) Addition $\vec{a}+\vec{b}$
2) cross Product $\vec{a}\times\vec{b}$
3) Dot Product $\vec{a}\cdot\vec{b}$
4) Scalar Triple Product $\vec{a}\cdot(\vec{b}\times\vec{c})$
5) Vector Product $(\vec{a}\times(\vec{b}\times\vec{c}) = (bac-cab))$

Vector Calculus
displacement — velocity - acceleration - jerk - joule - crackle - pop

Vector Calculus

Differentiation
- Gradient ∇
- Divergence $\vec{\nabla}\cdot$
- Curl $\vec{\nabla}\times$
- Laplacian ∇^2

Integration
- Gauss
- Stokes's
- Green

Vector Differential Operator = ∇ = del or nabla

Point Functions (position functions)

- Point functions are functions that are defined by the point in space.
- A variable quantity whose values at any point in a region is a function of position in space.

1) Scalar point function
   - equipotential surfaces (potential is scalar)
   - $\phi(x,y,z)$ is the function
   - if $\phi(x,y,z) = c$, level surfaces
   - isotherms, isobars
   - every point in region associated with a scalar value.

2) Vector point function
   - at every point, can associate a vector
   - $\vec{E}$, $\vec{B}$, gravitational field
   - $\vec{f}(x,y,z)$

Vector differential opesator defined as

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$

Applying ∇ onto a scalar, vector (dot & cross)

Point function: A variable quantity whose values at any point in a region depends on the position of the point

Scalar point function: if for each point in a region there corresponds a scalar function $\phi(x,y,z)$, then ϕ is called a scalar point function and the region is called a scalar field.

Vector point function: if for each point in a region there exists a vector function $\vec{f}(x,y,z)$, then $\vec{f}$ is called a vector point function and the region is called a vector field.

SPF: temperature distribution, potential

VPF: gravitational field, $\vec{E}$, $\vec{B}$

Level surfaces

$\phi(x,y,z) = c$ is called a level surface if every point on the surface can be associated with the same scalar.

eg: isotherm, isobar, equipotential surface

Vector Differential operator

The vector differential operator is denoted by ∇ (del or nabla) and is defined as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Change in a vector-valued function can be calculated only by parametrisation

i.e., if $\vec{r} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

$$d\vec{r} = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$$

Also, If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

GRADIENT

When ∇ is applied onto a scalar function $\phi(x,y,z)$

$\phi(x,y,z)$ is a scalar function

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\nabla\phi$ is called the gradient of the scalar function ϕ.

Geometrical Meaning of $\nabla\phi$ or grad ϕ

- if $\phi(x,y,z) = c$ is a level surface, then $\nabla\phi$ is a vector which is normal to the surface at any given point.

In other words, $\nabla\phi$ is perpendicular to the tangential vector

$$\phi(x,y,z) = c$$
$$d\phi = 0$$
$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$$
$$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = 0$$
$$\nabla\phi \cdot \vec{dr} = 0$$
$$\nabla\phi \perp \vec{dr}$$

- if $\phi(x,y,z)$ is not constant,
  $\nabla\phi$ = ~~directional derivative~~
  $DD = \nabla\phi \cdot \hat{a}$ = directional derivative

<u>Directional derivative</u> of a scalar point function $\phi(x,y,z)$ along x-axis is given by $\frac{\partial\phi}{\partial x}$.
Hence, $\nabla\phi$ gives the DD of ϕ along the coordinate axes.

For any given vector $\vec{a}$, the DD can be found by calculating $\nabla\phi \cdot \hat{a}$ (measure of steepness)

Angle of intersection b/w two surfaces

$$\cos\theta = \hat{\nabla\phi_1} \cdot \hat{\nabla\phi_2}$$

Max. dir. derivative is along $\nabla\phi$
Magnitude $= |\nabla\phi|$

For level surface, $\nabla\phi$ gives unit normal

Note!
for any scalar point function $\phi(x,y,z)$, grad ϕ is a vector that points in the direction of greatest increase of the function (greatest change) and it is 0 at the local extrema.

Significance of grad

1. For a level surface $\phi(x,y,z) = c$, grad $\hat{\phi}$ is the unit outward drawn normal. $(\hat{\nabla\phi})$

2. For level surfaces $\phi_1(x,y,z) = c_1$ and $\phi_2(x,y,z) = c_2$, the angle of intersection can be found using $\cos\theta = \hat{\nabla\phi_1} \cdot \hat{\nabla\phi_2}$

3. For any given surface $\phi(x,y,z)$, $\nabla\phi$ gives the change of the function in the direction of a vector.
   Directional derivative along $\vec{a}$ $= \nabla\phi \cdot \hat{a}$

   Max. ~~dd~~ DD $= |\nabla\phi| = \nabla\phi \cdot \hat{\nabla\phi}$

4. Given $\nabla\phi$, we can find a level surface for which $\nabla\phi$ is the outward normal.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|r|^2 = x^2+y^2+z^2$$

Diff. partially wrt x, y and z

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

* For cyclic functions (such as

$$\phi = x^3y^2 + y^3z^2 + z^3x^2$$

$$\nabla\phi = \sum_{\substack{i,j,k \\ x,y,z}} \frac{\partial\phi}{\partial x}\hat{i} = \sum(3x^2y^2 + 2xz^3)\hat{i}$$

$$\nabla\phi = (3x^2y^2+2xz^3)\hat{i} + (3y^2z^2+2yx^3)\hat{j} + (3z^2x^2+2zy^3)\hat{k}$$

1. If $f = 2xz^4 - x^2y$, find ∇f and $|\nabla f|$ at $(2,-2,-1)$

$$f = 2xz^4 - x^2y$$

$$\nabla f = (2z^4 - 2xy)\hat{i} + (-x^2)\hat{j} + (8xz^3)\hat{k}$$

At $(2,-2,-1)$

$$\nabla f = 10\hat{i} - 4\hat{j} - 16\hat{k}$$

$$|\nabla f| = 2\sqrt{93} = 19.29 = \sqrt{372} \text{ units}$$

2. $f = (x^2+y^2+z^2)\, e^{-\sqrt{x^2+y^2+z^2}}$, find ∇f

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{df}{dr}\frac{\partial r}{\partial x}\hat{i} + \frac{df}{dr}\frac{\partial r}{\partial y}\hat{j} + \frac{df}{dr}\frac{\partial r}{\partial z}\hat{k}$$

$$f = r^2 e^{-r}$$

$$\frac{df}{dr} = 2re^{-r} + r^2(-1)e^{-r}$$

$$= 2re^{-r} - r^2e^{-r} = e^{-r}(2r - r^2)$$

$$\nabla f = e^{-r}(2r-r^2)\left[\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right]$$

$$= e^{-r}(2-r)\left[x\hat{i}+y\hat{j}+z\hat{k}\right]$$

$$\nabla f = e^{-\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right)\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$\nabla f = \left(2-\sqrt{x^2+y^2+z^2}\right)\left(e^{-\sqrt{x^2+y^2+z^2}}\right)\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$\nabla f = (2-r)e^{-r}\vec{r}$: can be left in this form

3. Find the unit normal to the surface $(x-1)^2+y^2+(z+2)^2=9$ at the point $(3,1,-4)$

Level surface, unit normal $\hat{n} = \dfrac{\nabla f}{|\nabla f|}$

$$f = (x-1)^2+y^2+(z+2)^2 = 9$$

$$\nabla f = 2(x-1)\hat{i} + 2y\hat{j} + 2(z+2)\hat{k}$$

At $(3,1,-4)$, check if on sphere.

$(3-1)^2+1^2+2^2 = 4+4+1 = 9 = 9$ ✓

$$\nabla f = 2(2)\hat{i} + 2\hat{j} + 2(-2)\hat{k}$$

$$= 4\hat{i}+2\hat{j}-4\hat{k}$$

$$|\nabla f| = 6$$

$$\therefore \boxed{\hat{n} = \frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}-\frac{2}{3}\hat{k}}$$

4. What is the directional derivative of $f = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x \ln z - y^2 = -4$ at $(-1,2,1)$

~~normal to~~ $\phi = x \ln z - y^2 = -4$

at $(-1,2,1)$

$-\ln 1 - 4 = -4 \Rightarrow -4 = -4$ ✓.

$\therefore$ lies on surface

normal to $\phi = x \ln z - y^2 = -4$

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$; $\nabla\phi = (\ln z)\,\hat{i} - 2y\,\hat{j} + \dfrac{x}{z}\,\hat{k}$

at $(-1,2,1)$

$(\ln 1)\,\hat{i} - 4\hat{j} + \dfrac{-1}{1}\,\hat{k}$

$\nabla\phi = -4\hat{j} - \hat{k}$

$|\nabla\phi| = \sqrt{17}$

$\hat{n} = \dfrac{1}{\sqrt{17}}(-4\hat{j} - \hat{k})$

Scanned with CamSca

$DD = \nabla f \cdot \hat{n}$

$\nabla f = y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$

$= (1)\hat{i} + (2\times 2\times -1 + 1)\hat{j} + 3\times -1\times 1\,\hat{k}$

$\nabla f = \hat{i} - 3\hat{j} - 3\hat{k}$

$DD = (\hat{i} - 3\hat{j} - 3\hat{k})\cdot(-4\hat{j} - \hat{k})\left(\frac{1}{\sqrt{17}}\right)$

$DD = \frac{(12+3)}{\sqrt{17}} = \frac{15}{\sqrt{17}}$

5. Find the direction in which temperature changes most rapidly with the distance from the point (1,1,1) and determine the max rate of change if ~~$F(x,y,z) = xy +$~~ $T(x,y,z) = xy + yz + zx$ ~~and find the magnitude of that change.~~

$\nabla T(1,1,1) = ?\ |\nabla T|$

$T(1,1,1) = 1+1+1 = 3$

$\nabla T = \cancel{\sum \frac{\partial xy}{\partial x}} = \sum (y+z)\hat{i}$

$\nabla T = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

$\nabla T(1,1,1) = \boxed{2\hat{i} + 2\hat{j} + 2\hat{k}}$

$|\nabla T| = \sqrt{12} = \boxed{2\sqrt{3}}$

6. Find the angle of intersection between the level surfaces $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z=47$ at $(4,-3,2)$

$\phi_1 = x^2+y^2+z^2 = 29$
$\phi_2 = x^2+y^2+z^2+4x-6y-8z = 47$

$\phi_1(4,-3,2) = 29$

$\phi_2(4,-3,2) = 47.$

$\cos\theta = \hat{\nabla\phi_1} \cdot \hat{\nabla\phi_2}$

$\nabla\phi_1 = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$
$= 8\hat{i} - 6\hat{j} + 4\hat{k}$

$\nabla\phi_2 = (2x+4)\hat{i} + (2y-6)\hat{j} + (2z-8)\hat{k}$
$= 12\hat{i} - 12\hat{j} - 4\hat{k}$

$\cos\theta = \dfrac{(8\hat{i}-6\hat{j}+4\hat{k})\cdot(12\hat{i}-12\hat{j}-4\hat{k})}{\sqrt{8^2+6^2+4^2}\,\sqrt{12^2+12^2+4^2}}$

$\cos\theta = \dfrac{152}{2\sqrt{29}\,4\sqrt{19}} = \dfrac{19}{\sqrt{29}\,\sqrt{19}} = \sqrt{\dfrac{19}{29}}$

$\theta = 35.96^\circ$

7. Determine the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

$$\phi(x,y,z) = xy - z^2 = 0$$
$$\phi(4,1,2) = 4-4 = 0$$
$$\phi(3,3,-3) = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$\nabla\phi(4,1,2) = \hat{i} + 4\hat{j} - 4\hat{k}$$
$$\nabla\phi(3,3,-3) = 3\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\cos\theta = \cancel{\nabla\phi_1 \cdot \nabla\phi} \; \frac{\nabla\phi(4,1,2)\cdot\nabla\phi(3,3,-3)}{|\nabla\phi(4,1,2)|\,|\nabla\phi(3,3,-3)|}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}} = -\frac{\sqrt{22}}{22}$$

$$\theta = 102.3099$$
$$\boxed{\theta = 102.31^\circ}$$

Q. Find the values of a and b such that the surface $ax^2y + bz^3 = 4$ intersects the surface $5x^2 = 2yz + 9x$ orthogonally at $(1,-1,2)$

$$\phi_1 = ax^2y + bz^3 = 4$$
$$\phi_1(1,-1,2) = -a + 8b = 4 \longrightarrow (1)$$

$$\phi_2 = 5x^2 = 2yz + 9x$$
$$\phi_2(1,-1,2) = 5 = -4 + 9 = 5$$

$$\cos\theta = 0 = \nabla\phi_1 \cdot \nabla\phi_2$$

$$\nabla\phi_1 = 2axy\,\hat{i} + ax^2\,\hat{j} + 3bz^2\,\hat{k}$$
$$= -2a\hat{i} + a\hat{j} + 12b\hat{k}$$

$$\nabla\phi_2 = (9 - 10x)\hat{i} + 2z\hat{j} + 2y\hat{k}$$
$$= -\hat{i} + 4\hat{j} - 2\hat{k}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$2a + 4a - 24b = 0$$
$$6a - 24b = 0$$
$$a - 4b = 0$$
$$\boxed{a = 4b} \longrightarrow (2)$$

(2) in (1),

$$-4b + 8b = 4$$
$$4b = 4$$

$\boxed{b = 1}$ $\boxed{a = 4}$

9. If $\text{grad } f = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$, find $f(x,y,z)$ given $f(1,-2,2)=4$

$$\frac{\partial f}{\partial x} = 2xyz^3 \quad \cancel{\Rightarrow f = x^2yz^3}$$

$$\frac{\partial f}{\partial y} = x^2z^3 \quad \cancel{\Rightarrow f =}$$

$$\frac{\partial f}{\partial z} = 3x^2yz^2$$

$$\int_{y,z \text{ const}} \frac{\partial f}{\partial x}\,dx = \int 2xyz^3\,dx$$

$$f = x^2yz^3 + h(y,z) \rightarrow (1)$$

$$\int_{x,z \text{ const}} \frac{\partial f}{\partial y}\,dy = \int x^2z^3\,dy$$

$$f = x^2y.z^3 + g(x,z) \rightarrow (2)$$

$$\int_{x,y \text{ const}} \frac{\partial f}{\partial z}\,dz = \int 3x^2yz^2\,dz$$

$$f = x^2yz^3 + l(x,y) \rightarrow (3)$$

(1), (2), (3) : functions are constants

$$f = x^2yz^3 + c$$

$$f(1,-2,2) = -16 + c = 4$$

$$c = 20 \Rightarrow \boxed{f = x^2yz^3 + 20}$$

10. Find the scalar point function $\phi(x,y,z)$ such that $\vec{F} = \nabla\phi$ given ~~P.O~~

$$\vec{F} = (x+y+z)\hat{i} + (x+2y-z)\hat{j} + (x-y+2z)\hat{k}$$

$$\nabla\phi = (x+y+z)\hat{i} + (x+2y-z)\hat{j} + (x-y+2z)\hat{k}$$

$$\frac{\partial f}{\partial x} = x+y+z$$

$$\phi = \int_{y,z\ \text{const}} \frac{\partial\phi}{\partial x}\,dx = \frac{x^2}{2} + yx + zx + g(y,z)$$

$$\frac{\partial\phi}{\partial y} = x+2y-z.$$

$$\phi = \int_{x,z\ \text{const}} \frac{\partial\phi}{\partial y}\,dy = xy + y^2 - zy + h(x,z)$$

$$\frac{\partial\phi}{\partial z} = x-y+2z$$

$$\phi = \int_{x,y\ \text{const}} \frac{\partial\phi}{\partial z}\,dz = xy - yz + z^2 + l(x,y)$$

$\therefore \cancel{\phi = xy}$

$$g(y,z) = -2y + y^2 + z^2$$

$$h(x,z) = zx + \frac{x^2}{2} + z^2$$

$$l(x,y) = x^2 + y^2$$

$$\therefore \phi = \frac{x^2}{2} + yx + zx + y^2 + z^2 - yz$$

$$\phi(x,y,z) = xy - yz + zx + \frac{x^2}{2} + y^2 + z^2$$

11. Find the values of the constants a, b, c such that the DD of $f = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude of 64 in a direction parallel to z-axis.

$$DD = \nabla f \cdot \hat{k} = 64$$

$$\nabla f = (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

$$\nabla f = (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$$

∇f is along $\hat{k} \Rightarrow |\nabla f| = 64.$

$$\nabla f \cdot \hat{i} = 0 \Rightarrow 4a + 3c = 0$$
$$\nabla f \cdot \hat{j} = 0 \Rightarrow 4a - b = 0$$
$$\nabla f \cdot \hat{k} = 64 \Rightarrow 2b - 2c - 64 = 0$$

$$\boxed{a = 6,\ b = 24,\ c = -8}$$

$$\nabla f = 0\hat{i} + 0\hat{j} + 64\hat{k}$$

$$|\nabla f| = 64$$

12. $\nabla(3r^2 - 4\sqrt{r} + 6r^{-1/3})$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} ; \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad ; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$f = 3r^2 - 4\sqrt{r} + 6r^{-1/3}$$

$$\nabla = \frac{df}{dr}\frac{\partial r}{\partial x}\hat{i} + \frac{df}{dr}\frac{\partial r}{\partial y}\hat{j} + \frac{df}{dr}\frac{\partial r}{\partial z}\hat{k}$$

$$\nabla = \left(6r - \frac{4}{2\sqrt{r}} - \frac{1}{3} \times 6r^{-4/3}\right)\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right)$$

$$\nabla = \left(6 - \frac{2\sqrt{r}}{r^2} - \frac{2}{r^2 r^{1/3}}\right)\vec{r}$$

DIVERGENCE AND CURL

∇ operated on vector field.

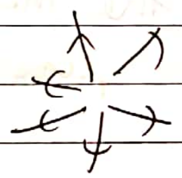
DIVERGENCE

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$\nabla \cdot \vec{F}$

>0



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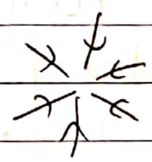

sink

$=0$

solenoidal

$\nabla \cdot \vec{F} = 0$

$\vec{F} = \nabla \times \vec{V}$

$\nabla \cdot (\nabla \times \vec{V}) = 0$

(STP)

If vector $\vec{F}$ is equal to $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is any vector point function, then ∇ applied onto $\vec{F}$ using the dot product is called the divergence of $\vec{F}$ and is denoted by $\nabla \cdot \vec{F}$ or div $\vec{F}$, which is a scalar.

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Divergence is the outflow of flux from a small, closed surface area (per unit volume) as volume ~~shrinks~~ shrinks to 0.

The divergence measures sources or drains of flow.

If div $\vec{F}$ is +ve, at a point, then that point acts as a source.

If div $\vec{F}$ is -ve at a point, then that point is a sink.

If div $\vec{F} = 0$, then $\vec{F}$ is called a solenoidal vector field and if it represents the velocity of a fluid, then $\nabla \cdot \vec{F} = 0 \Rightarrow$ the fluid is incompressible

Physical Interpretation of Divergence

Consider a vector field $\vec{F}$ that represents the velocity of a fluid.

Then, $\vec{\nabla} \cdot \vec{F}$ at a point is the measure of rate at which the fluid is flowing away or towards that point

Note: if a vector field $\vec{F}$ is solenoidal, then there exists another vector field $\vec{v}$ such that $\vec{F} = \vec{\nabla} \times \vec{v}$

CURL

If $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is any vector point function, then curl $\vec{F}$ or $\vec{\nabla} \times \vec{F}$ is a vector obtained by operating ∇ onto F using cross-product.

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i}$$

Geometric ~~Repre~~ Interpretation

Rigid body rotating at $\vec{\omega} = \omega\hat{k}$

$\vec{\omega} = \omega\hat{k}$

$\vec{v} = \vec{r} \times \vec{\omega}$

curl $\vec{v} = -2\vec{\omega}$

$$\vec{v} = \hat{i}(y\omega) - \hat{j}(x\omega)$$

$$\vec{\nabla} \times \vec{v} = -2\omega\hat{k}$$

~~Geometrical Interpretation~~

If F represents the velocity of ~~the~~ a fluid, then curl $\vec{F}$ measures the rotation of the fluid, or it represents the angular velocity at any point on the vector field.

If there is no rotation of the fluid anywhere in the vector field, then

$$\vec{\nabla} \times \vec{F} = 0$$

and the vector field $\vec{F}$ is called an irrotational vector field.

eg: whirlpool, tornado

Note: if a vector field $\vec{F}$ is irrotational, then there exists a scalar potential ϕ such that $F = \nabla\phi$

$$\nabla \times (\nabla\phi) = 0 \quad \text{(mixed partial}$$

LAPLACIAN OPERATOR (∇^2)

for a scalar point function f, div(grad(f)) or $\nabla \cdot (\nabla f)$ or $\nabla^2 f$ is called the Laplacian of f.

$$\nabla \cdot (\nabla f) = \nabla^2 f$$

Laplace equation : $\nabla^2 f$
heat equation, wave equation

PROPERTIES OF VECTOR DIFFERENTIAL OPERATOR

1. $\nabla(fg) = f(\nabla g) + g(\nabla f)$

2. $\nabla(f \pm g) = \nabla f \pm \nabla g$

3. $\nabla(cf) = c(\nabla f)$

4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

5. $\nabla\cdot(\phi\vec{F}) = \phi\,\nabla\cdot\vec{F} + \vec{F}\cdot\nabla\phi$ ☆

6. $\nabla\times(\phi\vec{F}) = \phi(\nabla\times\vec{F}) + \vec{F}\times(\nabla\phi)$ ☆

VECTOR OPERATOR IDENTITIES

1. $\text{div}(\text{grad}\,\phi) = \nabla\cdot(\nabla\phi) = \nabla^2\phi$ = scalar

2. $\text{curl}(\text{grad}\,\phi) = \nabla\times(\nabla\phi) = \vec{0}$ = vector

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \sum i\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) = 0.$$

☆

3. $\text{grad}(\text{div}\,\vec{F}) = \nabla(\nabla\cdot\vec{F}) = \nabla\times(\nabla\times\vec{F}) + \nabla^2\vec{F}$ = vector

4. $\text{curl}(\text{curl}(\vec{F})) = \nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot F) - \nabla^2 F$ = vector

13. Calculate $\nabla\cdot(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at (1,2,3)

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(5xy^2) + \frac{\partial}{\partial z}(xyz^3)$$

$$= 6x + 25xy + 3xyz^2$$

$$= 6 + 20 + 54 = 80$$

14. Determine the curl of the vector valued function $\vec{F} = xyz^2\hat{i} + yzx^2\hat{j} + zxy^2\hat{k}$ at (1,2,3).

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & yzx^2 & zxy^2 \end{vmatrix} = \sum i(2zxy - yx^2$$

$$= (2zxy - yx^2)\hat{i} + (2xyz - zy^2)\hat{j} + (2yzx - xz^2)\hat{k}$$

$$= 10\hat{i} + 0\hat{j} + 3\hat{k}$$

$$\nabla\times\vec{F} = 10\hat{i} + 3\hat{k}$$

15. Find $\nabla^2 f$ at the point $(2,3,1)$ when $f = \frac{xy}{z}$

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

$$= \nabla \cdot \left(\frac{y}{z}\hat{i} + \frac{x}{z}\hat{j} - \frac{xy}{z^2}\hat{k} \right)$$

$$= 0 + 0 + \frac{2xy}{z^3} = \frac{+2\times 2\times 3}{1} = +12.$$

~~at (2~~

$\nabla^2 f = +12.$ $\cancel{\nabla^2 = \frac{\partial^2}{\partial x^2}}$

16. Prove that $\frac{1}{r}$ satisfies the laplace equation.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$\cancel{\frac{1}{r}} = \cancel{\frac{r}{r^2}} =$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right)$$

$$= \frac{-1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) = \frac{-1}{r^3}(\vec{r})$$

$$\nabla\cdot\left(\nabla\left(\frac{1}{r}\right)\right) = \left(\frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{-z}{r^3}\right)\right)$$

$$\cancel{= \sum\left(\frac{\frac{-1}{r^3} + \frac{3x}{r^4}\frac{dr}{dx}}{r^6}\right) = \sum\frac{-\frac{r^2}{r^5} + \frac{3x^2}{r^5}}{r^6}}$$

$$\cancel{= \sum\frac{r^2+3x^2}{r^{11}} = \frac{-r^2+3r^2}{r^{11}}} = \frac{4}{r^9}$$

$$\nabla\cdot\left(\nabla\frac{1}{r}\right) = \sum \frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) \cancel{\frac{\partial r}{\partial x}}$$

$$\sum \frac{-1}{r^3} + (-x)\left(\frac{-3}{r^4}\right)\left(\frac{x}{r}\right)$$

$$= \frac{-3}{r^3} + \frac{3(x^2+y^2+z^2)}{r^5} = \frac{-3}{r^3} + \frac{3}{r^3} = 0.$$

11. The velocity vector of a fluid is given by

$$\vec{v} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}.$$

Is the motion of the fluid irrotational? If yes, find the velocity potential and check if the fluid is incompressible

such that $\nabla\phi = \vec{v}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = \sum(1-1)\hat{i} = \vec{0}$$

$\therefore$ irrotational fluid. (solenoidal field)

$\Rightarrow$ $\nabla\phi = \vec{v}$ for some ~~vector~~ scalar ϕ

$\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

~~$\nabla\phi = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$~~ $\quad \frac{(1)}{r^3} - (x)\left(\frac{-3x}{r^5}\right)$

$\phi =$ some function

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\frac{\partial\phi}{\partial x} = y+z \Rightarrow \phi = \int \frac{\partial\phi}{\partial x}\,dx = yx + zx + g(y,z)$$

y, z const

$$\phi = yx + zx + g(y,z)$$
$$\phi = zy + xy + h(x,z)$$
$$\phi = xz + yz + l(x,y)$$

$\phi = xy + yz + zx$

$\therefore$ velocity potential $\phi = xy + yz + zx$

$\nabla \cdot \vec{V} = 0+0+0 = 0 \Rightarrow$ fluid is incompressible

18. Prove that $\vec{A} = (2x^2+8xy^2z)\hat{i} + (3x^3y-3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}$ is not solenoidal but $\vec{B} = xyz^2(\vec{A})$ is solenoidal

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i}$$

$$= (-8yz^2 + 0)\hat{i} + ($$

$$\nabla \cdot \vec{A} = 4x + 8y^2z + 3x^3 - 3x - 8y^2z - 2x^3$$

$$\nabla \cdot \vec{A} = x + x^3 \; ; \; f = xyz^2$$

$$\nabla \cdot \vec{B} = \nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$$

$$= xyz^2(x + x^3) + \vec{A} \cdot (yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k})$$

$$= x^2yz^2 + x^4yz^2 + yz^2(2x^2 + 8xy^2z) + (xz^2)(3x^3y - 3xy) + 2xyz(-4y^2z^2 - 2x^3z)$$

$$= \cancel{x^2yz^2} + \cancel{x^4yz^2} + 2x^2yz^2 + \cancel{8xy^3z^3} + 3x^4yz^2 - \cancel{3x^2yz^2} - \cancel{8xy^3z^3} - 4x^4yz^2 = 0$$

19\. Determine the constant b such that $\vec{A} = (bx^2y + yz)\hat{i} + (xy^2 \cancel{\phi} - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ has 0 divergence.
(solenoidal vector field)

$\nabla \cdot \vec{A} = 2bxy + 2xy + 2xy = 0$

$2b + 4 = 0$

$\boxed{b = -2}$

$\vec{A} = \vec{\nabla} \times \vec{v}$ ~~where $\vec{v}$ is scalar potential~~

f.

20\. Find DD of $(\nabla \cdot \vec{u})$ at the point $(4,4,2)$ in the direction of the normal to the sphere $x^2 + y^2 + z^2 = 36$ where $\vec{u}$ is
$\vec{u} = xz\hat{i} + yz\hat{j} + zy\hat{k}$.

normal to sphere $\phi = x^2 + y^2 + z^2 = 36$

$\cancel{\nabla\phi}$ $\cancel{\phi = \nabla \cdot \vec{u}}$, $\hat{a} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$, $\cancel{|\nabla\phi| =}$
$= 8\hat{i} + 8\hat{j} + 4\hat{k}$, $|\nabla\phi| = \cancel{4\sqrt{5}}\ 12$

$\nabla \cdot \vec{u} = (z + x + y) = f$

$\nabla f = \hat{i} + \hat{j} + \hat{k}$

$DD = \cancel{\dfrac{\nabla f \cdot \nabla\phi}{4\sqrt{5}}} = \cancel{\dfrac{20}{4\sqrt{5}}}\ 12 \cancel{= \sqrt{5}} = \dfrac{5}{3}$

If the level surface is a sphere $x^2+y^2+z^2=a^2$, then the unit normal is

$$\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

normal to sphere: $\nabla\phi$

$$\frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{6}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{6} = \frac{4\hat{i}+4\hat{j}+2\hat{k}}{6}$$

$\nabla\cdot\hat{u} = (z+x+y) = f$

$\nabla f\cdot\hat{n} = DD.$

$$\nabla f = \frac{(\hat{i}+\hat{j}+\hat{k})\cdot(4\hat{i}+4\hat{j}+2\hat{k})}{6}$$

$$= \frac{10}{6} = \frac{5}{3}$$

21. Evaluate $\nabla \times \left(\frac{\vec{r}}{r^n}\right)$. If it is 0, find the scalar potential ϕ such that $-\nabla\phi$ is $\frac{\vec{r}}{r^n}$

$\nabla \times \left(\frac{\vec{r}}{r^n}\right)$. Find ϕ. $\phi(a) = 0$, $a > 0$.

$$\nabla \times \frac{\vec{r}}{r^n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^n} & \frac{y}{r^n} & \frac{z}{r^n} \end{vmatrix}$$

$$= \cancel{\sum (0-0)\hat{i} = \vec{0}}$$

$$\cancel{\therefore \frac{\vec{r}}{r^n} = \nabla\phi}$$

$$= \sum \frac{\partial}{\partial y}\left(\frac{z}{r^n}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^n}\right)$$

$$= \sum \frac{d}{dr}\left(\frac{z}{r^n}\right)\cdot\frac{y}{r} - \frac{d}{dr}\left(\frac{y}{r^n}\right)\frac{z}{r}$$

$$= \sum \frac{-n}{r^{n+2}} yz + \frac{n}{r^{n+2}} yz = \vec{0}$$

$$\cancel{\therefore \frac{\vec{r}}{r^n} - \nabla\phi = \frac{-\vec{r}}{r^{10}}}$$

$$\cancel{r^{10}\vec{r} + r^n\vec{r} = 0 \Rightarrow \vec{r}(r^{10} + r^n) = 0}$$

$$\cancel{r^n = -r^{10} \Rightarrow r^{n-10} = -1}$$

$$\cancel{r^{n-10}} \quad \cancel{= -1}$$

$\therefore \dfrac{\vec{r}}{r^n}$ is irrotational.

$$\cancel{\frac{\vec{r}}{r^n} = \nabla\phi}$$

But $-\nabla\phi = \dfrac{\vec{r}}{r^{10}}$

$$\frac{\partial\phi}{\partial x} = \frac{-x}{r^n} = \frac{-x}{(x^2+y^2+z^2)^{n/2}}$$

$$t = x^2+y^2+z^2$$
$$dt = 2x.$$

$$\int_{y,z\text{ const}} \frac{\partial\phi}{\partial x}\, dx = \int \frac{-dt}{2\, t^{n/2}}$$

if $n \neq 2$, $= \dfrac{-1}{2} \dfrac{t^{\left(\frac{-n}{2}+1\right)}}{\left(\frac{-n}{2}+1\right)}$

if $n=2$, $= \displaystyle\int \frac{-dt}{2t} = \frac{-1}{2} \ln t$

$$\phi = -\frac{1}{2}\frac{(x^2+y^2+z^2)^{(1-\frac{n}{2})}}{(1-\frac{n}{2})} + f(y,z)$$

for $n \neq 2$

for. $n=2$, $\phi = -\frac{1}{2}\ln(x^2+y^2+z^2) + f(y,z)$.

$$\phi = -\frac{1}{2}\frac{r^{2(1-n/2)}}{(1-n/2)} + f(y,z) \quad , n \neq 2$$

$$\phi = \frac{r^{2-n}}{n-2} + C \quad , n \neq 2$$

$$\phi = -\ln r + f(y,z) = -\ln r + C, \; n=2.$$

$\phi(a) = 0$

$(n \neq 2)$

$$\phi(a) = \frac{a^{2-n}}{n-2} + C = 0.$$

$$\boxed{C = \frac{a^{2-n}}{2-n}}$$

$n \neq 2$.

$$\phi = \frac{r^{2-n}}{n-2} + \frac{a^{2-n}}{2-n} \Rightarrow \boxed{\phi = \frac{r^{2-n} - a^{2-n}}{n-2}}$$

for $n=2$, $\phi = -\ln r + C$

$\phi(a) = 0 = -\ln a + C \Rightarrow C = \ln a$

$\boxed{\phi = \ln\frac{a}{r}}$ for $n=2$

Date
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~~vector~~ $\frac{\vec{r}}{r^n}$ is irrotational for all values of n

22. Find the value of n for which $r^n\vec{r}$ is solenoidal.

$$\nabla\cdot(r^n\vec{r}) = 0.$$

$$r^n\vec{r} = (x\hat{i}+y\hat{j}+z\hat{k})r^n$$

$$\nabla\cdot(r^n\vec{r}) = \sum\frac{\partial}{\partial x}(xr^n)$$

$$= \sum\left(r^n + x\,nr^{n-1}\frac{x}{r}\right)$$

$$= \sum r^n + x^2 nr^{n-2}$$

$$= 3r^n + nr^{n-2}(r^2)$$

$$\nabla\cdot r^n\vec{r} = 3r^n + nr^n = (3+n)r^n$$

$$(3+n)r^n = 0. \Rightarrow \boxed{n=-3}$$

$$(r^n \neq 0)$$

$n\cancel{=-3}$

23. Determine the constants a, b and c such that $\vec{A} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

Also, find a scalar function f such that $\nabla f = \vec{A}$.

$$\nabla \times \vec{A} = \sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i}$$

$$\nabla \times \vec{A} = (c+1)\hat{i} + (a-4)\hat{j} + (b-2)\hat{k}$$

$$a = 4, \; b = 2, \; c = -1$$

$$\nabla f = \vec{A} \Rightarrow \frac{\partial f}{\partial x} = x + 2y + 4z$$

$$\frac{\partial f}{\partial y} = 2x - 3y - z$$

$$\frac{\partial f}{\partial z} = 4x - y + 2z$$

$$f = \frac{x^2}{2} + 2xy + 4zx + g(y,z) \quad -yz + z^2 - \frac{3y^2}{2}$$

$$= 2xy - \frac{3y^2}{2} - zy + h(x,z)$$

$$= 4xz - yz + z^2 + l(x,y)$$

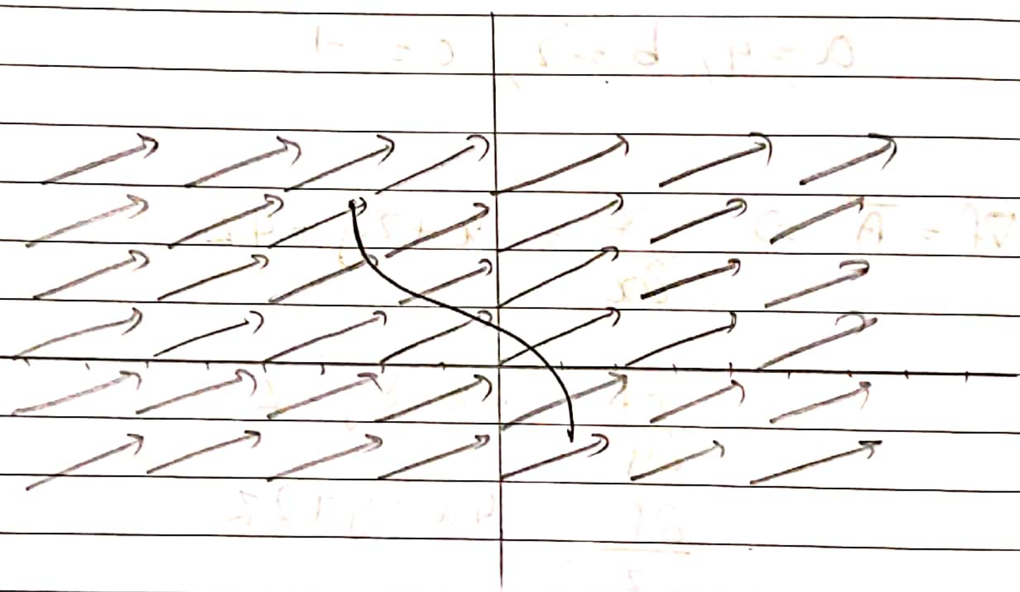
$$f = \frac{x^2 - 3y^2 + 2z^2}{2} + 2xy + 4zx - yz$$

Vector Integration
- Green's — 2D
- Stoke's — 3D
- Gauss's — relates double/triple divergence

Line Integral

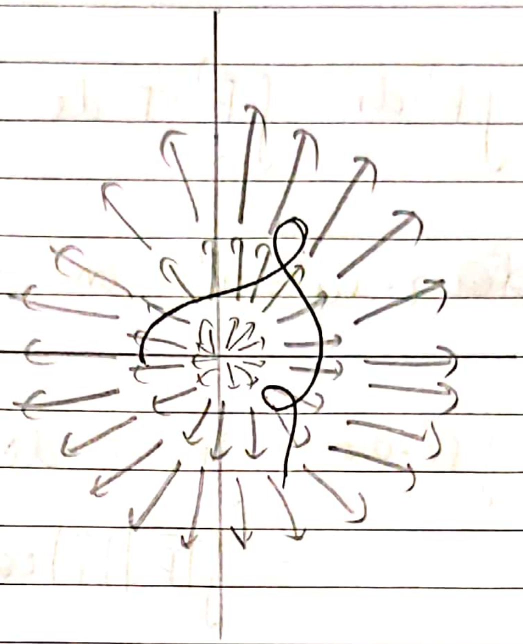
Vector Fields

$2\hat{i}+\hat{j}$ - constant



$x\hat{i}$

$x\hat{i} + y\hat{j}$ $\quad \nabla \cdot \vec{F} = 2 \implies$ source



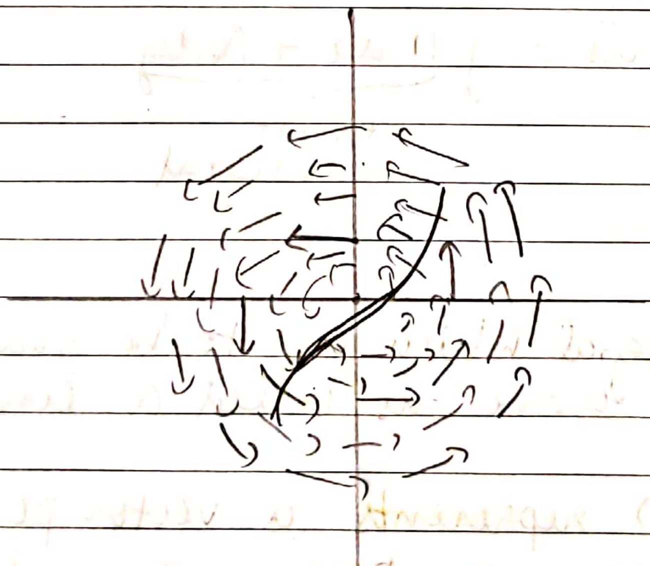
$-y\hat{i} + x\hat{j}$ $\quad \nabla \cdot \vec{F} = 0 \implies$ solenoidal

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$\sum \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i}$$

$$= \hat{k}(2)$$

$$= 2\hat{k}$$



velocity

circulation (angiogram) $\int \vec{v} \cdot d\vec{r}$

How much the vector field is aligned along the line - line integral

$$\int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \hat{T} \, ds$$

for $x^2 + y^2 = a^2$ on $-y\hat{i} + x\hat{j} = \vec{F}$

$|\vec{F}| = \sqrt{x^2 + y^2}$

$$\int \vec{F} \cdot d\vec{r} = \oint \vec{F} \cdot \hat{T} \, ds$$

$$= \int |\vec{F}| |\hat{T}| \cos\theta \, ds = \int a \, ds = 2\pi a^2$$

$$\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\int \vec{F} \cdot d\vec{r} = \int \underbrace{M dx + N dy}_{\text{exact integral}}$$

Any integral which is to be evaluated along a curve is called a line integral

If $\vec{F}(P)$ represents a vector point function defined at every point on a curve c, then

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(P_i) \cdot d\vec{r_i}$$

If the path of the integral is a closed curve, then the ~~vector field is dento~~. line integral ~~is~~ is denoted by

$$\oint_C \vec{F} \cdot d\vec{r}$$

(positive orientation ↺ ; change sign if arrow clockwise)

If $\vec{F}$ represents the force field, then $\int_C \vec{F} \cdot d\vec{r}$ is the total work done in moving an object along the curve c.

If $\vec{F}$ represents the velocity of the fluid and c is a closed contour, the $\int \vec{F} \cdot d\vec{r}$ gives the circulation of the fluid around c the closed curve.

If $\vec{F}$ represents an electric field, then $\int \vec{F} \cdot dr$ around a loop is equal to the voltage generated around the loop.

$\int \vec{F} \cdot d\vec{r}$ measures how much the vector field is aligned with the curve.

Example: $\vec{F} = x\hat{i} + y\hat{j}$, $C: x^2+y^2=a^2$

$$\int \vec{F}\cdot d\vec{r} = \int_C (x\hat{i}+y\hat{j})\cdot(dx\,\hat{i}+dy\,\hat{j})$$

$$= \int x\,dx + y\,dy$$

Line integral can be evaluated only by parametrisation

$x = a\cos\theta$ $\quad dx = -a\sin\theta\, d\theta$
$y = a\sin\theta$ $\quad dy = a\cos\theta\, d\theta$

$$= \int_0^{2\pi} -a^2\cos\theta\sin\theta\,d\theta + a^2\cos\theta\sin\theta\,d\theta$$

$= 0.$

$\vec{F} = -y\hat{i} + x\hat{j}$ $\quad C: x^2+y^2=a^2$

$$\int \vec{F}\cdot d\vec{r} = \int -y\,dx + x\,dy$$

$$= \int_0^{2\pi} +a^2\sin^2\theta\,d\theta + a^2\cos^2\theta\,d\theta = 2\pi a^2$$

Line integral depends only on curve.

conservative vector field $\longrightarrow$ curl is 0.

For conservative vector fields, $\oint_C \vec{F} \cdot d\vec{r} = 0$
(closed curve)

$$\int_a^b \vec{F} \cdot d\vec{r} \quad \text{where } \nabla \times \vec{F} = 0.$$

$$= \int_a^b \vec{\nabla \phi} \cdot \vec{dr} = \int_a^b \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= \int_a^b d\phi = \phi(b) - \phi(a)$$

1. If $\vec{F}$ is a conservative vector field ($\nabla \times \vec{F} = 0$) and C is a closed curve, then $\oint_C \vec{F} \cdot d\vec{r} = 0$

If $\vec{F}$ is a conservative vector field as C is a curve joining the points a and b, then $\int_a^b \vec{F} \cdot d\vec{r} = \phi(b) - \phi(a)$ where $\vec{\nabla \phi} = \vec{F}$

Note: in a conservative vector field, the line integral is arc independent

2. A line integral can be solved only by parametrisation

3. A line integral depends only on the path but not on the parametrisation.

24. Find the total work done in moving a particle in a force field $A = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$W = \int_C A \cdot d\vec{r}$$

$$\int \vec{A} \cdot d\vec{r} = \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$x = t^2+1$ —— $dx = 2t\,dt$

$y = 2t^2$ —— $dy = 4t\,dt$

$z = t^3$ —— $dz = 3t^2\,dt$

$$= \int_1^2 3(t^2+1)(2t^2)(2t\,dt) - 5(t^3)(4t\,dt) + 10(t^2+1)(3t^2)\,dt$$

$$= \int_1^2 12(t^5+t^3) - 20(t^4) + 30(t^4+t^2)\,dt$$

$W = 303$

25. Evaluate $\int_c \vec{A}\cdot d\vec{r}$ where $\vec{A} = 2x\hat{i} + 4y\hat{j} - 3z\hat{k}$

A and c is the curve whose position vector is $\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ where t varies from 0 to π.

$$\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$$
$$d\vec{r} = -\sin t\,dt\,\hat{i} + \cos t\,dt\,\hat{j} + dt\,\hat{k}$$

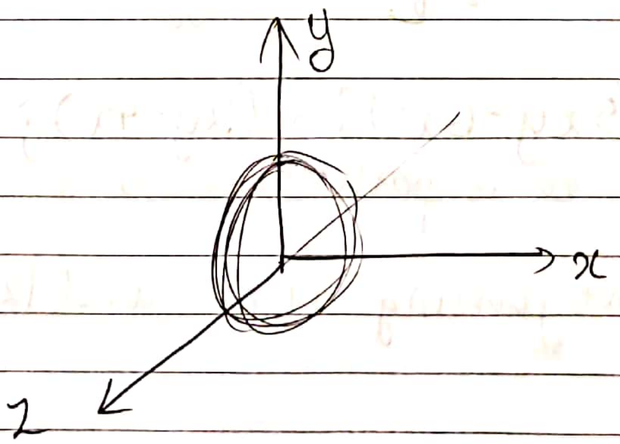
$$W = \int_c \vec{A}\cdot d\vec{r} = \int_0^{\pi} 2(\cos t)(-\sin t\,dt) + 4(\sin t)(\cos t\,dt) - 3t\,dt$$

$$= \int_0^{\pi} (2\sin t\cos t - 3t)\,dt = \int_0^{\pi} 2\sin t\cos t\,dt - \frac{3\pi^2}{2}$$

$$= \int_0^{\pi} \sin(2t)\,dt - \frac{3\pi^2}{2} = \left[\frac{-\cos 2t}{2}\right]_0^{\pi} = -\frac{3\pi^2}{2}$$

$= -$

26. Evaluate $\int \vec{A}\cdot d\vec{r}$ given $\vec{A} = y\hat{i} + z\hat{j} + x\hat{k}$ and where c is the curve $c = y^2 + z^2 = 1$ and $x = 0$



$x = 0$
$y = \cos\theta$
$z = \sin\theta$

$dx = 0$
$dy = -\sin\theta\,d\theta$
$dz = \cos\theta\,d\theta.$

Scanned with CamSca

$\int \vec{A} \cdot d\vec{r} = ?$

$$\text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \sum \hat{i}(-1) = -\hat{i} - \hat{j} - \hat{k}$$

$$\int \vec{A} \cdot d\vec{r} = \int_0^{2\pi} (\cos\theta)\, 0 + (\sin\theta)(-\sin\theta\, d\theta) + 0$$

$$= -\int_0^{2\pi} \sin^2\theta\, d\theta = -\frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{1}\, d\theta.$$

$$= -\frac{1}{2}\left(\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 2\theta\, d\theta\right)$$

$$= -\frac{2\pi}{2} + \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\pi.$$

27. If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, P.T. $\int_c \vec{F} \cdot d\vec{r}$ ~~is~~ is path-dependent for the curve joining $(1,1)$ and $(2,8)$

Note: cannot prove $\vec{\nabla} \times \vec{F} \neq 0$ and say ~~force~~ work is path independent.

Two possible paths:

$$(y-1) = \left(\frac{8-1}{2-1}\right)(x-1)$$

$$y-1 = 7x-7$$

$$C_1: \quad y = 7x-6 \longrightarrow (1)$$

$$dy = 7\,dx$$

$$C_2: \quad y = x^3 \longrightarrow (2)$$

$$dy = 3x^2\,dx$$

(Parametrisation)

$$W_1 = \int_{C_1} \vec{F}\cdot\vec{dr} = \int_1^2 ((5x)(7x-6)-6x^2)\,dx + ((2)(7x-6)-4x)(3x^2)\,dx$$

$$= \int_1^2 (35x^2-30x-6x^2)\,dx + (14x-12-4x)(7)\,dx$$

$$= \frac{131}{3}$$

$$\cancel{= \int_1^2 (29x^2-30x)\,dx + (30x^3-36x^2)\,dx}$$

$$\cancel{= \int_1^2 -7x^2+30x^3-30x\,dx = \frac{307}{6}}$$

$$W_2 = \int_{C_2} \vec{F}\cdot\vec{dr} = \int_1^2 (5x^4-6x^2)\,dx + (2x^3-4x)(3x^2\,dx)$$

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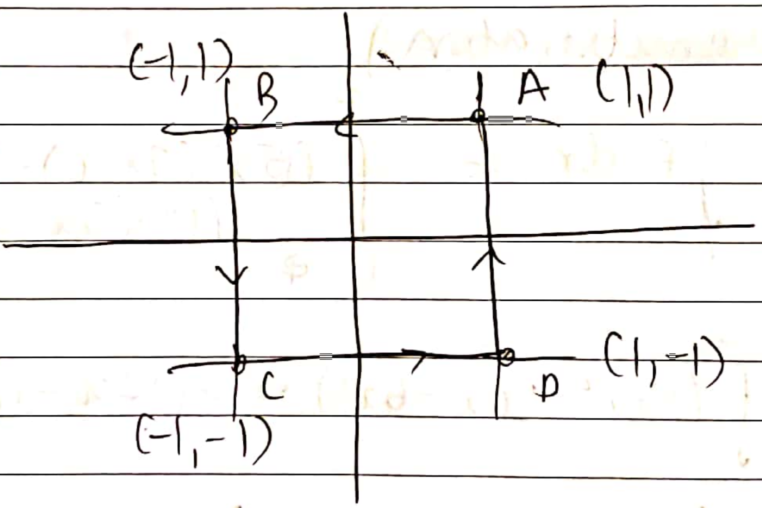
$$W_2 = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^2 (5x^4-6x^2)\,dx + 3x^2(2x^3-4x)\,dx$$

$$W_2 = \int_1^2 5x^4-6x^2+6x^5-12x^3\,dx = 35$$

28. Evaluate the line integral $\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where c is the square formed by the lines $y=\pm 1,\ x=\pm 1$.

must walk such that enclosed area is to our left.



direction: ABCDA.

$$\text{curl } \vec{F} = \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \hat{k}\,(2x - x) = x\hat{k} \neq \vec{0}.$$

$\therefore\ \vec{F}$ is non-conservative.

$$= \oint_C \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CD} \vec{F}\cdot\vec{dr} + \int_{DA} \vec{F}\cdot\vec{dr}$$

$$= \int_1^{-1} x^2 + x\,dx + \int_1^{-1} \cancel{1+y^2\,dy} + \int_{-1}^{1} x^2 - x\,dx + \int_{-1}^{1} \cancel{+1+y^2\,dy}$$

$y=1,\ dy=0$ $\quad x=-1,\ dx=0$ $\quad y=1,\ dy=0$ $\quad x=1,\ dx=0$

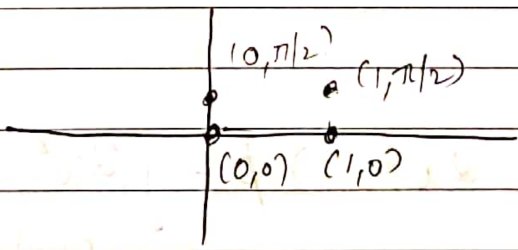
$x: -1$ to 1.

$$= \int_1^{-1} \cancel{x^2} + x - \cancel{x^2} + x\,dx = \int_1^{-1} 2x\,dx = \left[x^2\right]_1^{-1} = 0.$$

$\therefore$ line integral $= 0$.

This is a special case of a non-conservative field with a closed loop integral of 0.

29. Find the circulation of the fluid whose velocity vector is $e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ around the rectangle whose vertices are $(0,0)$, $(1,0)$, $(1,\pi/2)$, $(0,\pi/2)$



$$\text{curl } \vec{F} = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k} = (e^x\cos y - e^x \cos y)\hat{k}$$

$\text{curl } \vec{F} = \vec{0} \Rightarrow$ conservative field.

$\therefore$ for a closed loop, $\int \vec{F}\cdot\vec{dr} = 0$.

30. Find the constant a such that the vector field $\vec{V} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is conservative. Also, find the work done in moving a particle from $(1,2,-3)$ to $(1,-4,2)$ in this field.

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}\left((1-a)z^2 - 3z^2\right) + \hat{k}\left(2(a-2)x - ax\right)$$

$$1 - a = 3 \Rightarrow \boxed{a = 4}$$

scalar potential ϕ such that $\nabla\phi = \vec{V}$

$$\frac{\partial\phi}{\partial x} = 4xy - z^3$$

$$\phi = \int_{y,z \text{ const}} \frac{\partial\phi}{\partial x}\, dx = \int 4xy - z^3\, dx = 2x^2y - xz^3 + g(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2x^2 \Rightarrow \phi = \int_{x,z \text{ const}} 2x^2\, dy = 2x^2y + h(x,z)$$

$$\frac{\partial\phi}{\partial z} = -3xz^2 \Rightarrow \phi = \int_{x,y \text{ const}} -3xz^2 dz = \frac{-3xz^3}{3} + l(x,y)$$

$$\phi = 2x^2y - xz^3 + c$$

work done $= \phi(1,-4,2) - \phi(1,2,-3)$.

$$W = 2(1)(-4) - (1)(8) - ((2)(1)(2)) + (1)(-27) + \cancel{c} - \cancel{c}$$
$$= -8 - 8 - 4 - 27$$
$$W = -47$$

31. Find the work done in moving a particle once round an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ if the force field is given by $\vec{F} = (3x - 4y + 2z)\hat{i} + (4x + 2y - 3z^2)\hat{j} + (2xz - 4y^2 + z^3)\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \sum \hat{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)$$

$$= \hat{i}(-8y + 6z) + \hat{j}(2 - 2z) + \hat{k}(4 + 4)$$

$$= (-8y + 6z)\hat{i} + (2 - 2z)\hat{j} + 8\hat{k} \neq 0.$$

$\therefore$ non-conservative $\vec{F}$.

$$W = \int \vec{F} \cdot d\vec{r}$$

$x = 4\cos\theta$ $dx = -4\sin\theta \, d\theta$
$y = 3\sin\theta$. $dy = 3\cos\theta \, d\theta$
$z = 0$ $dz = 0$.

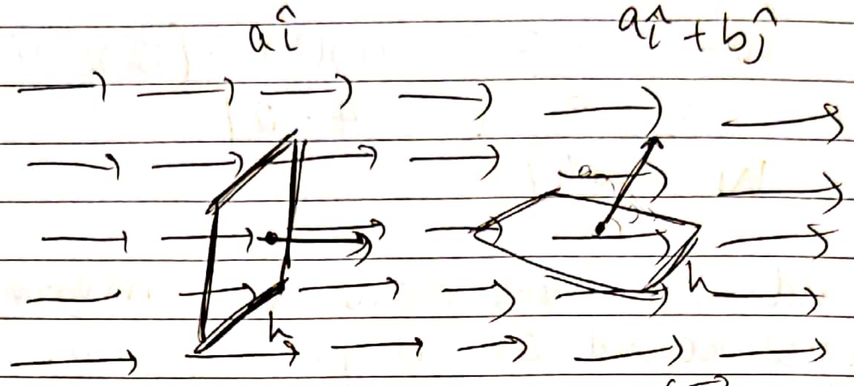
$$W = \int_0^{2\pi} (12\cos\theta - 12\sin\theta)(-4\sin\theta) + (16\cos\theta + 6\sin\theta)(3\cos\theta) \, d\theta$$

$$= \int -48\sin\theta\cos\theta + 48\sin^2\theta + 48\cos^2\theta + 18\sin\theta \cancel{d\theta}\cos\theta \, d\theta$$

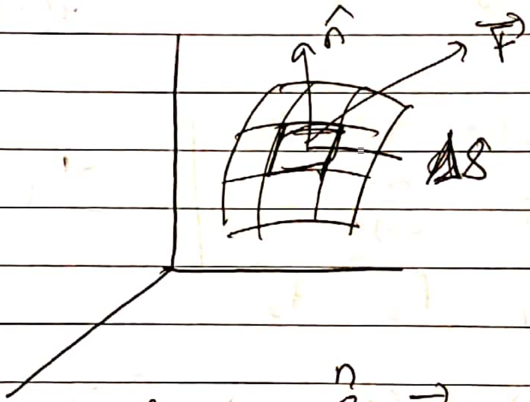
$$= \int_0^{2\pi} -30\sin\theta\cos\theta \, d\theta + 48 \times 2\pi = 96\pi.$$

Surface Integral

Flux.



$\text{flux} = a h^2$ $\quad\quad$ $\text{flux} = (|\vec{E}| \cos\theta)\, h^2$



$$\lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(P_i)\, \hat{n}\, \Delta S_i$$

$$= \iint_S \vec{F} \cdot \underbrace{\hat{n}\, ds}_{d\vec{s}} = \iint_S \vec{F} \cdot d\vec{s}$$

also sometimes

$$= \int_S \vec{F} \cdot d\vec{s} \quad ; \text{for closed surface:}$$

$$\oiint_S \vec{F} \cdot d\vec{s} \quad \text{(Gauss's law)}$$

Surface integral

Any integral that is evaluated over a surface is called a surface integral

If $\vec{F}(P_i)$ is a continuous vector point function defined at every point of a two-sided surface S, then ~~the~~

$$\lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(P_i) \cdot \hat{n} \, \Delta S_i$$

is called the normal surface integral of $\vec{F}(P_i)$ on the surface S, and is denoted by

$$\iint_S \vec{F} \cdot \hat{n} \, ds$$

$$= \iint_S \vec{F} \cdot \vec{ds} = \int_S \vec{F} \cdot \vec{ds}$$

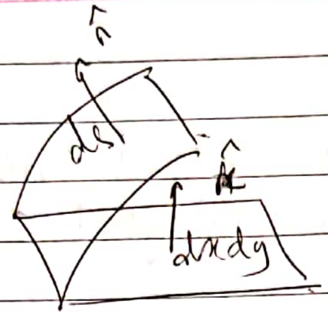
for closed surfaces, $\oiint_S \vec{F} \cdot \vec{ds}$ (Gauss' Law)

Evaluating a Surface Integral

Surface integral can be evaluated by projecting the surface onto one of the coordinate planes.

If the surface is projected onto the x-y plane, then ~~dxdy =~~

$$\cos\theta \, ds = dx\,dy$$



$$\hat{n}\cdot\hat{k} = \cos\theta$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

onto y-z plane: $ds = \dfrac{dy\,dz}{\hat{n}\cdot\hat{i}}$

onto x-z plane: $ds = \dfrac{dx\,dz}{\hat{n}\cdot\hat{j}}$

$$= \iint \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$$

32. Evaluate $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

and s is the surface of the sphere $x^2+y^2+z^2 = a^2$

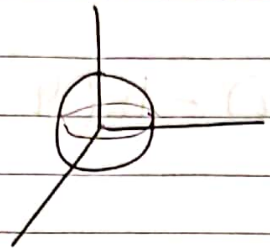
$$\hat{n} = \nabla S = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\vec{F}\cdot\hat{n} = \frac{x^2+y^2+z^2}{a} = a$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint a\,ds = \boxed{4\pi a^3}$$

or.



$x = r\cos\theta\sin\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\phi$

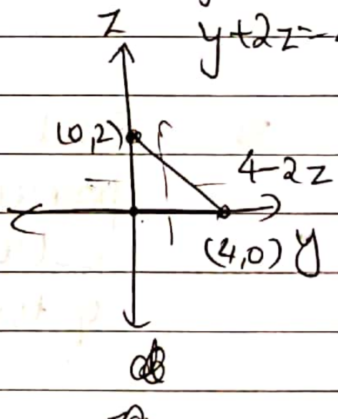
$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$

$$= \int_0^{2\pi}\int_0^{\pi} a a^2 \sin\phi \, d\phi \, d\theta$$

$$= a^3(2\pi)(-\cos\phi)_0^{\pi} = \boxed{4\pi a^3}$$

33. Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ ~~or~~ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.

$3y+6z=12$
$y+2z=4$



$$\hat{n} = \nabla\phi = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$\vec{A}\cdot\hat{n} = \frac{36z - 36 + 18y}{7}$$

$$ds = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} = \frac{dy\,dz}{2/7}$$

projected on y-z plane $(x=0)$.

$y: 0$ to $4-2z$
$z: 0$ to 2.

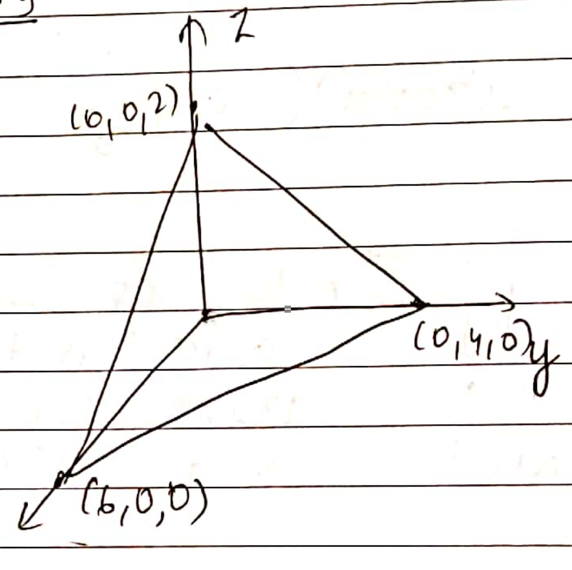
$$\iint_S \vec{A}\cdot\hat{n}\,ds = \int_{z=0}^{2}\int_{y=0}^{4-2z} \frac{36z-36+18y}{2}\,dy\,dz$$

$$= \int_0^2 \left[18yz - 18y + \frac{9y^2}{2}\right]_0^{4-2z} dz$$

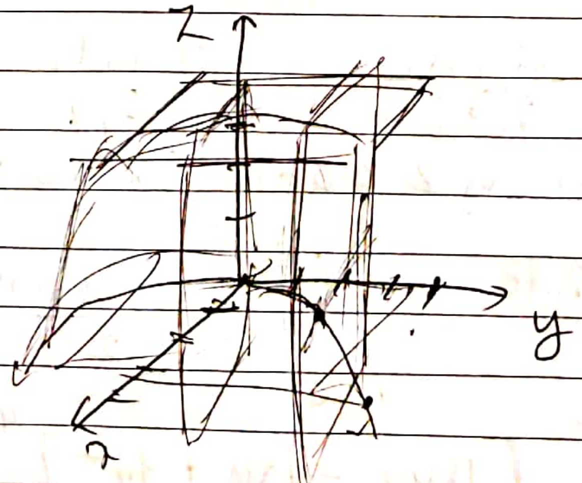
$$\int_0^2 18z(4-2z) - 18(4-2z) + \frac{9}{2}(4-2z)^2 \, dz$$

$= 24$

Geometry



34. Find the flux across the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y=4$, $z=6$ if the velocity vector $\vec{v} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$.



$\hat{n} = \nabla\phi = \nabla(y^2 - 8x)$

$$\hat{n} = \frac{-8\hat{i} + 2y\hat{j}}{\sqrt{64+4y^2}}$$

$$\vec{v}\cdot\hat{n} = \frac{-16y - 2yz}{\sqrt{64+4y^2}} = \frac{-8y - yz}{\sqrt{16+y^2}}$$

Projected onto y-z plane.

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dy\,dz}{8}\left(\sqrt{64+4y^2}\right)$$

limits: $y : 0 \to 4$, $z : 0 \to 6$.] shadow is rectangular.

$$\int_{z=0}^{6}\int_{y=0}^{4} \frac{(-8y-yz)2}{\cancel{\sqrt{64+4y^2}}}\,\frac{\cancel{(\sqrt{64+4y^2})}}{8}\,dy\,dz$$

$$= \int_0^6\int_0^4 \frac{\cancel{-8y}\;-16y - 2yz}{8}\,dy\,dz.$$

$$\cancel{= \int_0^6 \frac{-(16+2z)}{8}\,dz \int^4 y\,dy} \quad \cancel{= \int^6 2 + \frac{z}{4}\,dz}$$

$$\int_0^6 -2y\,dy$$

$$\cancel{= \left(-2(6) - \frac{1}{4}\left(\frac{36}{2}\right)\right)\left(\frac{16}{2}\right)}$$

$= 12 + \cancel{\frac{36}{8} - \frac{2\times16}{8}}$

$$\int_0^6 \frac{-16-2z}{8}\, dz \int_0^4 y\, dy$$

$$= -\frac{1}{8}\left(16\times6+36\right)\left(\frac{16}{2}\right) = 132 \text{ m}^3/\text{s}.$$

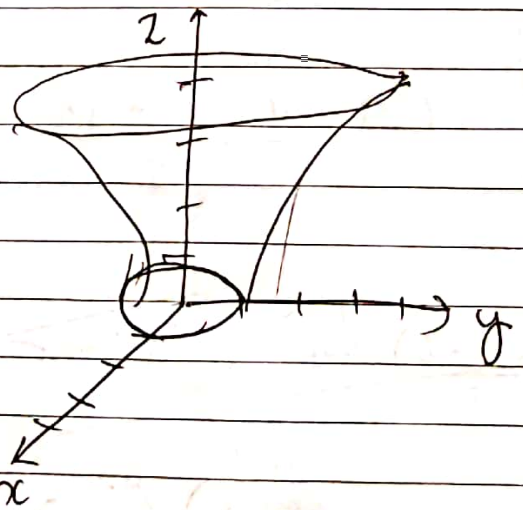
<u>NOTE</u>: cannot project onto x-y plane as $\hat{n}\cdot\hat{k} = 0$

35. Compute the flux of the vector field $\vec{A} = x\hat{i} + y\hat{j} + \sqrt{x^2+y^2-1}\,\hat{k}$ through the outer side of the hyperboloid of one sheet $z = \sqrt{x^2+y^2-1}$ bounded by the planes $z=0$ and $z=\sqrt{3}$.

$$z^2 = x^2+y^2-1$$

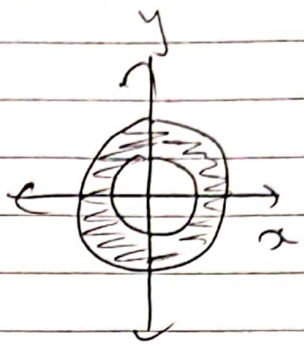
or

$$x^2+y^2-z^2 = 1$$



$$\hat{n} = \nabla\phi = \frac{\cancel{2}x\hat{i} + \cancel{2}y\hat{j} - \cancel{2}z\hat{k}}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$\vec{A}\cdot\hat{n} = \frac{x^2+y^2 - z\sqrt{x^2+y^2-1}}{\sqrt{x^2+y^2+z^2}}$$



Project onto x-y plane

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy\,\sqrt{x^2+y^2+z^2}}{z}$$

$$\vec{A}\cdot\hat{n} = \frac{x^2+y^2-(x^2+y^2-1)}{\sqrt{x^2+y^2+x^2+y^2-1}}$$

$$= \frac{1}{\sqrt{2x^2+2y^2-1}}$$

$$\vec{A}\cdot\hat{n}\,ds = \frac{dx\,dy}{\sqrt{2x^2+2y^2-1}}\;\frac{\sqrt{2x^2+2y^2-1}}{\sqrt{x^2+y^2-1}}$$

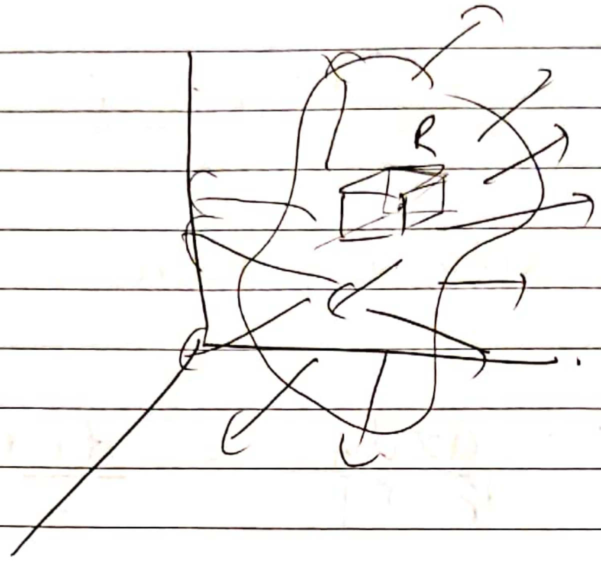
$$\vec{A}\cdot\hat{n}\,ds = \frac{dx\,dy}{\sqrt{x^2+y^2-1}}$$

$$\iint \vec{A}\cdot\hat{n}\,ds = \iint \frac{dx\,dy}{\sqrt{x^2+y^2-1}} = \int_{\theta=0}^{2\pi}\int_{r=1}^{2} \frac{r\,dr\,d\theta}{\sqrt{r^2-1}}$$

$$= (2\pi)\int_{1}^{2} \frac{r\,dr}{\sqrt{r^2-1}} \qquad r^2-1 = t. \qquad 2r\,dr = dt$$

$$= \pi\int_0^3 \frac{dt}{\sqrt{t}} = \pi\left[\frac{t^{1/2}}{1/2}\right]_0^3 = \boxed{2\pi\sqrt{3}}$$

Volume Integral



Any integral that is evaluated over the volume bounded by a solid, then the volume integral is the limit of the sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(P_i)\,\Delta V_i = \iiint_V \vec{F}\,dV$$

$$= \iiint_V f_1\hat{i}\,dV + \iiint_V f_2\hat{j}\,dV + \iiint_V f_3\hat{k}\,dV$$

Physical significance

If ρ is the density of a fluid with velocity $\vec{u}$, then the linear momentum is

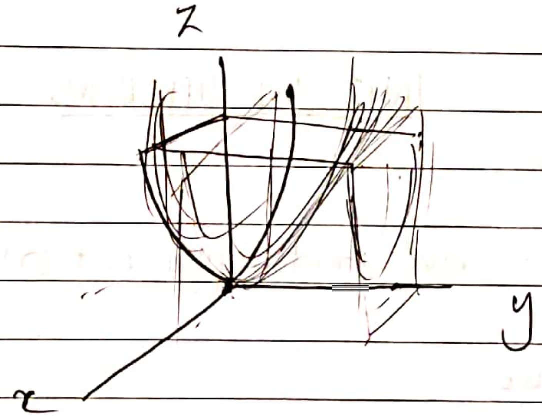
$$\iiint_V \rho\vec{u}\,dV$$

If $\nabla\phi$ represents the concentration gradient, then the total concentration is

$$\iiint \nabla\phi \, dV$$

36. Evaluate $\iiint_V \vec{B} \, dV$ where V is the region bounded by the surface $x=0$, $y=0$, $z=x^2$, $y=6$ and $z=4$. and $\vec{B} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$

$$\iiint 2xz\hat{i} - x\hat{j} + y^2\hat{k} \, dV$$



$$\int_{z=0}^{4}\int_{y=0}^{y=6}\int_{x=0}^{x=\sqrt{z}} 2xz\hat{i} - x\hat{j} + y^2\hat{k} \, dx\,dy\,dz$$

$$= \iint [x^2 z]_0^{\sqrt{z}}\hat{i} - \iint \left(\frac{x^2}{2}\right)_0^{\sqrt{z}}\hat{j} + \iint [x]_0^{\sqrt{z}} y^2\hat{k}$$

$$= \iint z^2\hat{i} - \iint \frac{z}{2}\hat{j} + \iint \sqrt{z}\, y^2\hat{k}$$

$$= \int_{0}^{4}\int_{6}^{6} z^2 \hat{i}\, dy\, dz - \int_{z=0}^{4}\int_{y=0}^{6} \frac{z}{2}\hat{j}\, dy\, dz + \int_{z=0}^{4}\int_{y=0}^{6} \sqrt{z}\, y^2 k\, dy\, dz$$

$$= \int_{0}^{4} 6z^2 \hat{i}\, dz - \int_{0}^{4} 3z\hat{j}\, dz + \int_{0}^{4} \frac{\sqrt{z}}{3} 6^3\, dz.$$

$$= \frac{6(4)^3\hat{i}}{3} - \frac{3(4^2)}{2}\hat{j} + \frac{6^3}{3}\, 4^{3/2} \times \frac{2}{3}$$

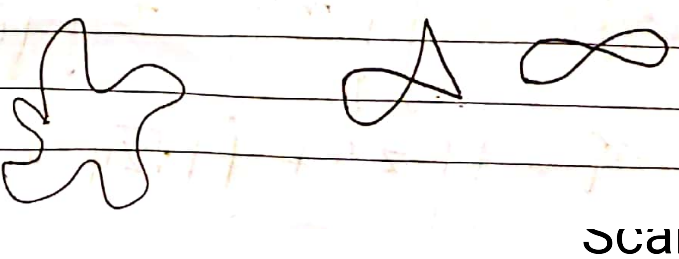
$$= 128\hat{i} - 24\hat{j} + 384\hat{k}$$

INTEGRAL THEOREMS

1) Green's – HW. find application
2) Stoke's
3) Divergence.

GREEN'S THEOREM

- ~~on~~ only 2-dimensions
- simple curve: if a line can be drawn intersecting the curve at more than 2 points. It is not simple.



not simple.

Green's theorem gives a relationship between a line integral around a simple closed curve C and the double integral over the region R enclosed by the curve C

Statement:

Let ~~..~~ C be a simple closed, positively-oriented, differentiable curve in 2D plane

$\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ be any VF which is continuous and differentiable, ~~decide that~~ defined at every point inside and on the boundary, then

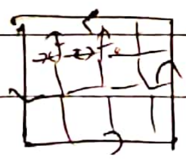
$$\oint \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k} \, dA$$

Where R is the region enclosed by the closed curve C.

$$\oint M dx + N dy = \iint (N_x - M_y) \, dy \, dx.$$

Intuition

If $\vec{F}$ represents the velocity of a fluid, then, Green's Theorem explains that the circulation of the fluid ~~is equ~~ around the boundary is equal to the microscopic circulation in the region R.



Applications

1) Green's Theorem can be used to find the area enclosed by a simple closed curve c and is given by

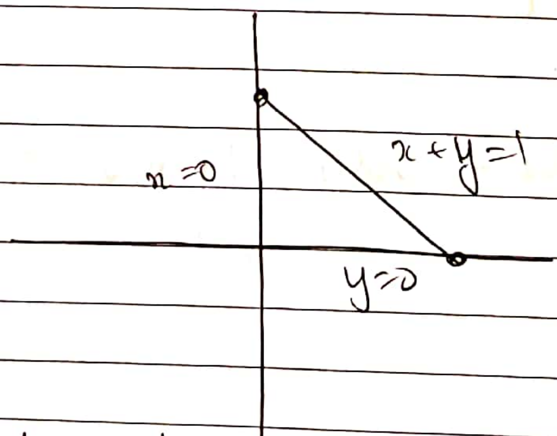
$$A = \iint dx\,dy = \oint_C -y\,dx = \oint_C x\,dy = \frac{1}{2}\oint -y\,dx + x\,dy$$

$$= \frac{1}{2}\oint_C (r(\theta))^2\,d\theta$$

$\Rightarrow -y\hat{i} + x\hat{j}$

37. Evaluate $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

where c is the boundary of the region defined by $x=0$, $y=0$ and $x+y=1$



$M = 3x^2 - 8y^2$
$M_y = -16y$
$N = 4y - 6xy$
$N_x = -6y$

Acc. to Green's Theorem,

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_0^1 \int_0^{1-x} (-6y + 16y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[5y^2 \right]_0^{1-x} dx = \int_0^1 5(1-x)^2\, dx$$

$$= -5 \left[\frac{(1-x)^3}{3} \right]_0^1 = \frac{+5}{3}\left(\frac{1}{1}\right) = \frac{5}{3}$$

without Green's:

$$\oint_{C_1} (3x^2 - y^2)\,dx + \oint_0^a (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 \cancel{-y^2})\,dx + \int_1^0 (3x^2 - (1-x)^2)\,dx + \dots$$

parameters.

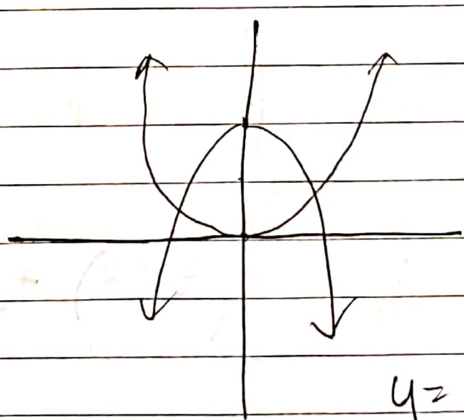
★ do again

38. Verify Green's theorem for the integral $\oint_C y^3 dx + 3xy^2 dy$ where C is boundary of the +vely oriented region bounded by $y = x^2$ and $y = 2 - x^2$

$$\oint y^3 dx + 3xy^2 dy$$

$$= \iint (N_x - M_y)\,dy\,dx$$

$$= 0.$$



$M = y^3$

$M_y = 3y^2$

$N = 3xy^2$

$N_x = 3y^2$

$y = x^2$

$dy = 2x\,dx$

$$\oint_C y^3 dx + 3xy^2 dy$$

$y = x^2$ $\quad$ $y = 2-x^2$

$dy = 2x dx$ $\quad$ $dy = -2x dx$

$$= \int_{-1}^{1} x^6 dx + \int_{1}^{-1} (2-x)^3 dx + \int_{-1}^{1} 3x^7 \, 2x \, dx$$

$(2/7)$ $\quad$ (-20) $\quad$ $(4/3)$

-1

$$+ \int_{1} 3x(2-x^2)^2 (-2x \, dx) \quad 28/5$$

$$= \cancel{2\int_0^1 x^6 dx} = \frac{2}{7} + \int_{1}^{-1} 8 - 12x + 6x^2 + x^3 \, dx$$

$$+ \int_{-1}^{1} 6x^8 dx + \int_{+1}^{-1} -6x^2(4+x^4-4x^2) \, dx$$

$\cancel{Q10}$

$$= \frac{2}{7} + \int_{1}^{-1} 8 + 6x^2 dx + 12\int_0^1 x^8 dx + \int_{-1}^{1} 6x^2(4+x^4-4x^2)$$

$$= \frac{2}{7} - 2\int_0^1 8 + 6x^2 dx + 12\int_0^1 x^8 dx + -\int_0^1 12x^2(4+x^4-4x^2)$$

$$= \frac{2}{7} - 2\left(8+2\right) + \frac{12}{9} - \frac{48}{3} - \frac{12}{7} - \frac{48}{5}$$

39. How much work is required to move an object in the vector field $\vec{F} = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$ along the path joining between the two points from A (0,0,0) to B(2,4,6)?

$$W = \int_a^b \vec{F} \cdot d\vec{r}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix}$$

$$= \hat{i}(x^2 - x^2) - \hat{j}(2xy - 2xy) + \hat{k}(2xz - 2xz)$$

$$= \vec{0}$$

$\therefore \vec{F}$ is conservative.

Scalar potential ϕ such that $\nabla\phi = \vec{F}$

$$\frac{\partial\phi}{\partial x} = 2xyz \Rightarrow \phi = x^2yz + g(y,z)$$

$$\frac{\partial\phi}{\partial y} = x^2z \Rightarrow \phi = x^2yz + h(x,z)$$

$$\frac{\partial\phi}{\partial z} = x^2y \Rightarrow \phi = x^2yz + l(x,y)$$

$$\therefore \phi = x^2yz + C.$$

work done $= \phi(B) - \phi(A) = (4)(4)(6) = 96$

38. (again)

$$\oint_C y^3 dx + 3xy^2\, dy$$

$$= \int_{-1}^{1} x^6 dx + 3x(x^4)(2x\,dx)$$

$$+ \int_{1}^{-1} (2-x^2)^3 dx + 3x(2-x^2)^2(-2x\,dx)$$

$$= 7\int_{-1}^{1} x^6 dx + \int_{1}^{-1} 8-12x^2+6x^4-x^6\, dx$$

$$+ \int_{1}^{-1} -6x^2(4-4x^2+x^4)\, dx$$

$$= 8\int_{-1}^{1} x^6 dx + \int_{1} 8-12x+6x^4 + \int_{1} -24x^2+24x^3 - 6x^5$$

$$= 8$$

40. Find the area of the loop of folium of Descartes $x^3+y^3=3axy$.

★

↳ check for symmetry (x-axis, y-axis, $x=\pm y$)

replace x & y.

$$y^3+x^3=3ayx \rightarrow \text{same curve}$$

↳ find location of loop.

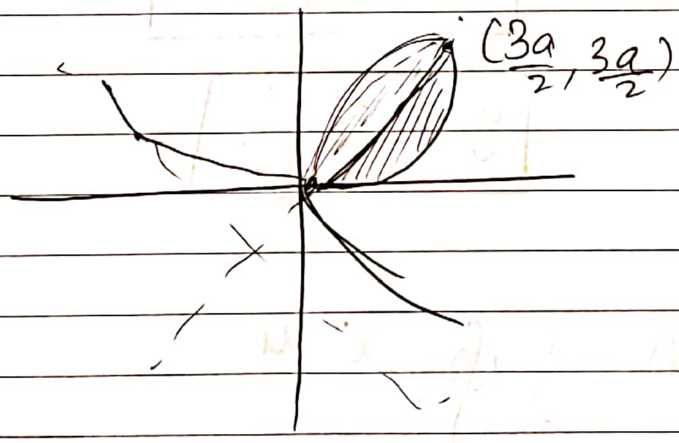
where does it intersect $x=y$?

$$2x^3=3ax^2$$

$x=0$ or $x=\frac{3a}{2}$

$(0,0)$ and $\left(\frac{3a}{2},\frac{3a}{2}\right)$

$\left(x=\frac{3at}{1+t^3}\right.$



$\left.y=\frac{3at^2}{1+t^3}\right)$

$$A=\frac{1}{2}\oint_C -y\,dx+x\,dy$$

if $y=xt$, $dy=x\,dt+t\,dx$

$$A = \frac{1}{2} \oint_C -xt\,dx + x(x\,dt + t\,dx)$$

$$= \frac{1}{2} \oint_C x^2\,dt$$

Replace $y = xt$ in $x^3 + y^3 = 3axy$

$$x^3 + t^3x^3 = 3ax^2t$$

$$x^3(1+t^3) = 3ax^2t$$

$$\boxed{x = \frac{3at}{1+t^3}}$$

$$\boxed{y = \frac{3at^2}{1+t^3}}$$

$$A = \frac{1}{2} \oint_C x^2\,dt$$

To find limits of t

use $y = x$: $\frac{\cancel{3a}t}{\cancel{1+t^3}} = \frac{\cancel{3a}t^2}{\cancel{1+t^3}}$

$t^2 - t = 0 \Rightarrow t = 0,\ t = 1$

t ~~var~~ varies from 0 to 1.

$$\text{area} = A = \frac{1}{2}\int_0^1 \frac{(3at)^2}{(1+t^3)^2}\,dt$$

$$= \frac{1}{2}\int_0^1 \frac{9a^2t^2}{(1+t^3)^2}\,dt$$

$s = 1+t^3$

$ds = 3t^2\,dt$

$$= \frac{1}{2}\int_1^2 \frac{3a^2\,ds}{s^2}$$

$t=0 \Rightarrow s=1$

$t=1 \Rightarrow s=2$

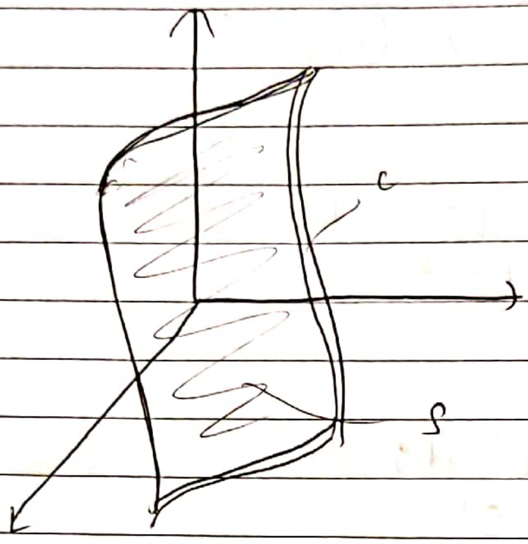
$$= \frac{3a^2}{2}\left(\frac{-1}{s}\right)_1^2 = \frac{3a^2}{2}\left(\frac{-1}{2}+1\right) = \frac{3a^2}{4}$$

total area of loop $= \frac{2\times 3a^2}{4} = \frac{3a^2}{2}$

(4).

STOKE'S THEOREM

extension of Green's
(Generalisation)



$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F} \cdot \hat{n}) \, ds$$

If $\vec{F}$ is a continuous differentiable vector point function, defined at every point inside and on an open surface bounded by a simple closed curve C, then

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F} \cdot \hat{n}) \, ds$$

eg: $x\hat{i} + y\hat{j} + z\hat{k}$ curl = 0

(or)

$\nabla \times (f(x)\hat{i} + f(y)\hat{j} + f(z)\hat{k}) = 0$

for curl to exist, coefficients of unit vectors should not be functions along their dir

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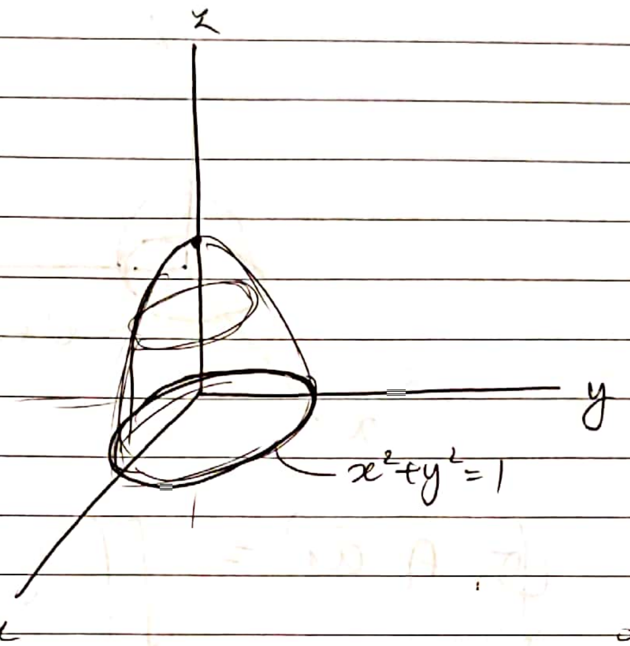
41. Evaluate $\iint \nabla \times (y\hat{i} + z\hat{j} + x\hat{k}) \cdot \hat{n}\, ds$ over the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$



$$= \oint_{x^2+y^2=1} \vec{F} \cdot \vec{dr} = \oint_{x^2+y^2=1,\ z=0} y\,dx + \cancel{z\,dy} + \cancel{x\,dz}$$

$dz = 0$

$$= \oint_{x^2+y^2=1} y\,dx$$

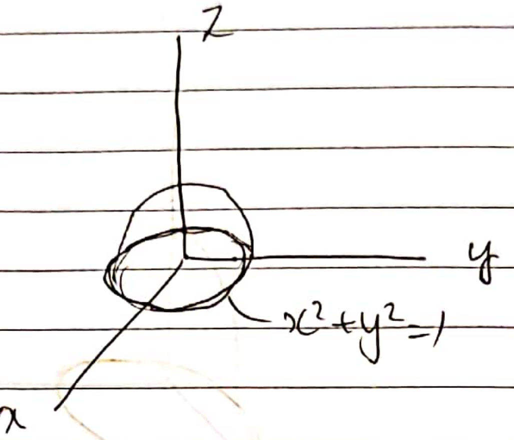
$y = \sin\theta$ $\quad dy = \cos\theta$

$x = \cos\theta$ $\quad dx = -\sin\theta$

$$= \int_{\theta=0}^{2\pi} -\sin^2\theta\, d\theta = \int_0^{2\pi} \frac{\cos 2\theta - 1}{2}\, d\theta$$

$\boxed{= -\pi}$

42. Verify Stoke's Theorem for the vector field $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half of the sphere $x^2+y^2+z^2=1$



$$\oint_C \vec{A}\cdot d\vec{r} = \iint_S (\nabla\times\vec{A}\cdot\hat{n})\,ds,$$

LHS:

$$\oint_{\substack{x^2+y^2=1\\ z=0}} (2x-y)dx - yz^2\cancel{dy}^{0} - y^2z\cancel{dz}^{0}$$

$y = \sin\theta \qquad dy = \cos\theta\, d\theta$

$x = \cos\theta \qquad dx = -\sin\theta\, d\theta$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\, d\theta)$$

$$= \int_0^{2\pi} -\sin 2\theta + \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \boxed{\pi}$$

RHS,

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\,(-2yz+2yz) - \hat{j}(0-0) + \hat{k}(+1)$$

$$\nabla\times\vec{A} = \hat{k}$$

$$\hat{n} = \nabla\,\text{sphere} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$= \iint_S z\,ds.$$

$$\cancel{ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}}$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \iint_S \cancel{z}\,\frac{dx\,dy}{\cancel{z}}$$

$$= \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \frac{(2\pi)}{2} = \boxed{\pi}$$

$\therefore$ LHS = RHS

43. Suppose S is a surface obtained by cutting the lightbulb at its base, and the base represents a unit circle $x^2+y^2=1$. Find the flux integral $\iint \text{curl } \vec{F} \cdot \hat{n}\, ds$ for the vector field

$$\vec{F} = (z^2-2z)x\hat{i} + (\sin(xyz)+y+1)\hat{j} + e^{z^2}\sin z^2\,\hat{k}.$$

Acc. to stoke's Theorem

$$\iint_S (\nabla \times F \cdot \hat{n})\, ds = \oint_C \vec{F}\cdot d\vec{r}$$

$$\oint_{x^2+y^2=1,\ z=0} (z^2-2zx)\,dx + (\sin(xyz)+y+1)\,dy + (e^{z^2}\sin z^2)\,dz$$

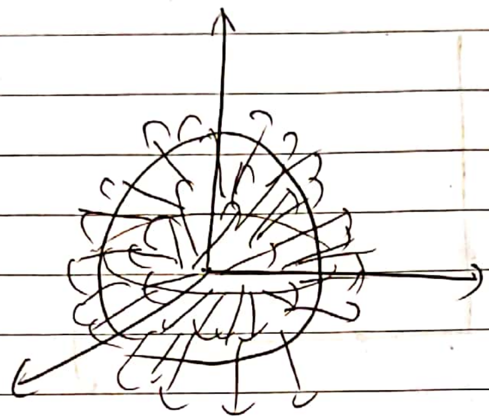
$$\oint_{x^2+y^2=1,\ z=0} (y+1)\,dy \qquad \begin{array}{l} y=\sin\theta \\ dy=\cos\theta\, d\theta \end{array}$$

$$= \int_0^{2\pi} (\sin\theta+1)\cos\theta\, d\theta = \int_0^{2\pi} \sin\theta\cos\theta\, d\theta$$

$$= \left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} = 0.$$

GAUSS' DIVERGENCE THEOREM

$$\oiint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV$$



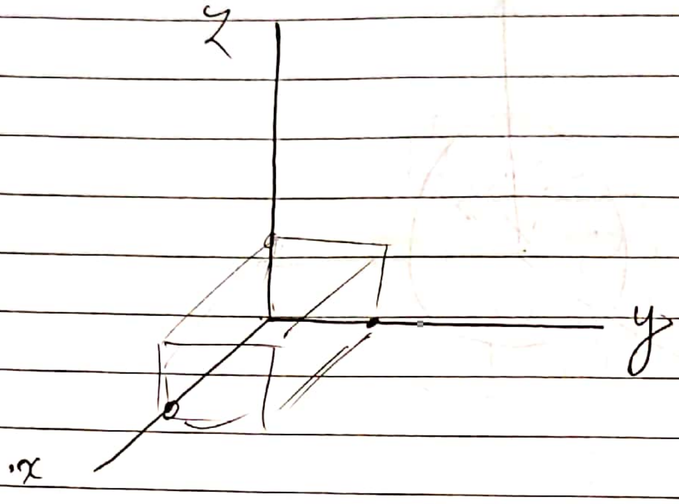
If $\vec{F}$ is a continuous, differentiable vector point function defined at every point inside and on the surface of a solid enclosing a volume V and bounded by a closed surface S, then

$$\oiint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV$$

44. Using Divergence Theorem evaluate the surface integral

$$\oiint_S \sin x \, dy\,dz + (2-\cos x)y \, dz\,dx$$

where S is a parallelopiped $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$,



$$\oiint \vec{F}\cdot\hat{n}\, ds = \iiint \operatorname{div}\vec{F}\cdot dV$$

$dydz \longrightarrow$ on y-z plane normal is $\hat{i}$

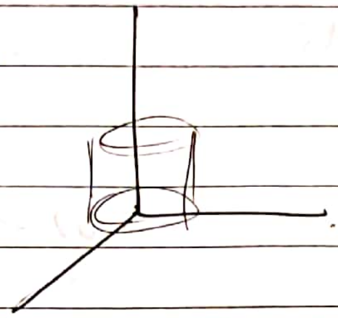
$$\vec{F} = \sin x\,\hat{i} + (2-\cos x)y\,\hat{j}$$

$$\operatorname{div}\vec{F} = \cancel{\cos x} + (2-\cancel{\cos x}) = 2$$

$$\iiint 2\,dV = 2\iiint dV = 2\times3\times2\times1$$

$$= \boxed{12}$$

45. Verify Gauss' Divergence Theorem for the function $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2+y^2=9$, $z=0$ and $z=2$.



$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$.

$$\oiint_S \vec{F}\cdot\hat{n}\, ds = \iiint \operatorname{div} F \cdot dV$$

$$\nabla\cdot\vec{F} = 2z$$

RHS:
$$\iiint 2z\, dV = \iiint 2z\rho\, d\rho\, d\phi\, dz$$

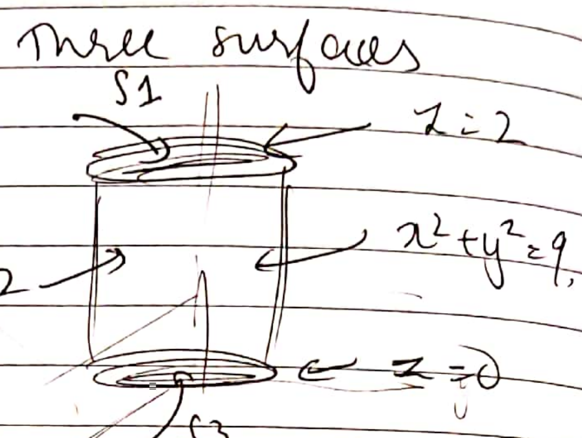
$$\int_{z=0}^{2}\int_{\rho=0}^{3}\int_{\phi=0}^{2\pi} 2z\rho\, d\phi\, d\rho\, dz$$

$$= (2\pi)\left(\frac{9}{2}\right)(\cancel{4}^{\,2}) = 36\pi$$

LHS:

$\hat{n}$: normal to cylinder.

$= \oiint_S \vec{F} \cdot \hat{n}\, ds$



$= \oiint_{S1} \vec{F} \cdot \hat{n}\, ds + \oiint_{S2} \vec{F} \cdot \hat{n}\, ds + \oiint_{S3} \vec{F} \cdot \hat{n}\, ds$

for S1, $\hat{n} = \hat{k}$; for S3, $\hat{n} = -\hat{k}$

for S2, $\hat{n} = \nabla S_2 = \dfrac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2+y^2}} = \dfrac{x\hat{i}+y\hat{j}}{3}$

~~LHS~~: for S1, $ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}} = dx\,dy$

$= \iint \vec{F} \cdot \hat{k}$ for S3, $ds = -dx\,dy$

for S2, $\cancel{ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}} = dx\,dy}$

$ds = \dfrac{dy\,dz}{\hat{n}\cdot\hat{i}} = \left(\dfrac{3\,dy\,dz}{x}\right)$

$\therefore$ LHS: $\quad \vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$

$$\iint_{S_1} z^2 \, dx\,dy + \iint_{S_2} \frac{2xy\cancel{3}dy\,dz}{\cancel{3}\,x} + \iint_{S_2} -z^2 dx\,dy$$

~~on z~~

$$\iint_{S_1} 4\,dx\,dy + \iint_{S_2} 2y\,dy\,dz + \cancel{\iint_{S_3} -0\,dx\,dy}$$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{3} 4r\,dr\,d\theta \times 0 + \cancel{\int_{z=0}^{2}\int_{y=3}^{3} 2y\,dy\,dz}$$

$$\cancel{= 8(2\pi)\left(\frac{9}{2}\right)\times 0 + 2\left(\frac{9}{2}\right)(2)}$$

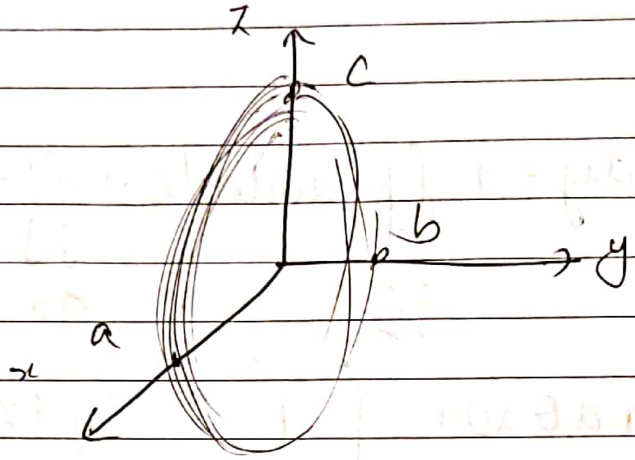
$$= (4)(2\pi)\left(\frac{9}{2}\right) + 0 = \boxed{36\pi}$$

RHS = LHS

46. Verify GDT for $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ taken over the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\oiint \vec{A}\cdot\hat{n}\,ds = \iiint \nabla\cdot\vec{A}\,dV$$

RHS:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$\nabla\cdot\vec{A} = 2x + 2y + 2z$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\cancel{\int^{c}\iiint (2x + 2y + 2z)\,dx\,dy\,dz}$$

$\cancel{z = -c \quad y =}$

We need to parameterise

$$x = a r \cos\theta \sin\phi$$
$$y = b r \sin\theta \sin\phi$$
$$z = c r \cos\phi$$

$$dV = abc\, r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$r : 0 \to 1$

$\theta : 0 \to 2\pi$

$\phi : 0 \to \pi$

$$\iiint 2(x+y+z)\,dV$$

$$= \iiint \sin\phi\, abc\, 2r^3 (a\cos\theta\sin\phi + b\sin\theta\sin\phi + c\cos\phi)\, dr\,d\phi\,d\theta$$

$$= \cancel{\iiint 2abc(}$$

$$= \iiint 2abc\, r^3 (a\sin^2\phi\cos\theta + b\sin\theta\sin^2\phi + c\cos\phi\sin\phi)\, dr\,d\phi\,d\theta$$

$$= 2abc\left[a\int_{r=0}^{1}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} r^3 \cancel{\cos\theta\sin^2\phi}^{\,0} + b\int_{r=0}^{1}\int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \cancel{r^3\sin\theta\sin\phi}^{\,0} + c\iiint \cos\phi\sin\phi\; dr\,d\phi\,d\theta\right]$$

$$= +2abc^2 \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \cancel{\cos\phi}\;\frac{\sin 2\phi}{2}\; d\phi\,d\theta\,dr$$

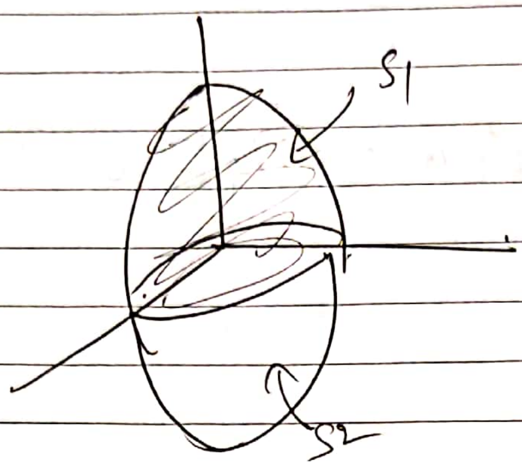
$$\alpha = 2\phi$$

$$\frac{d\alpha}{4} = \frac{d\theta}{2}$$

$$= 2abc^2 \int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{\alpha=0}^{2\pi} \frac{\sin\alpha}{4}\, d\alpha\,d\theta\,dr$$

$$= \boxed{0}$$

LHS:



2 surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\vec{A}=x^2\hat{i}+y^2\hat{j}+z^2\hat{k}$$

$$\oiint_S \vec{A}\cdot\hat{n}\,ds$$

$$\hat{n}_1=\nabla\phi_1=\frac{\frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}+\frac{2z}{c^2}\hat{k}}{2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{c^2}{z^4}}}$$

$$ds_1=\frac{dx\,dy}{\hat{n}\cdot\hat{k}}=\frac{dx\,dy}{\cancel{2}z}\,c^2\times\cancel{2}\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{c^2}{z^4}}$$

for S2, $\hat{n}_2=-\hat{n}_1$

$z=0$ on xy plane

$$x=ar\cos\theta$$
$$y=br\sin\theta$$
$$dx\,dy=ab\,r\,dr\,d\theta$$

$$\iint_{S_1} \vec{A}\cdot\hat{n}_1 \frac{dx\,dy}{\hat{n}_1\cdot\hat{k}} + \iint_{S_2} \vec{A}\cdot\hat{n}_2 \frac{dx\,dy}{\hat{n}_2\cdot\hat{k}}$$

$$\hat{n}_1 = \frac{\frac{x}{a^2}\hat{i} + \frac{y}{b^2}\hat{j} + \frac{z}{c^2}\hat{k}}{\sqrt{\sum \frac{x^2}{a^4}}} \; ; \; \hat{n}_2 = \frac{-\frac{x}{a^2}\hat{i} - \frac{y}{b^2}\hat{j} - \frac{z}{c^2}\hat{k}}{\sqrt{\sum \frac{x^2}{a^4}}}$$

$$ds_1 = \frac{dx\,dy}{z} c^2 \sqrt{\sum \frac{x^2}{a^4}} \; ; \; ds_2 = \frac{dx\,dy}{z} c^2 \sqrt{\sum \frac{x^2}{a^4}}$$

$$S1 : \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} \left(\frac{\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2}}{\cancel{\sqrt{\sum \frac{x^2}{a^4}}}}\right) \frac{abr\,dr\,d\theta}{z} c^2 \cancel{\sqrt{\sum \frac{x^2}{a^4}}}$$

$$\cancel{= \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} \left(\frac{x^3}{a^2} + \frac{y^3}{b^2} + z\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\right) \frac{abr\,dr\,d\theta}{z} c^2 \sqrt{\sum \frac{x^2}{a^4}}}$$

$$\cancel{S2 : \int_{r=0}^{1}\int_{\theta=\pi/2}^{0} \left(\frac{x^3}{a^2} + \frac{y^3}{b^2} + z\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\right) \frac{abr\,dr\,d\theta}{z} c^2 \sqrt{\sum \frac{x^2}{a^4}}}$$

$$\frac{z^2}{c^2} = 1 - r^2$$

$$z^2 = c^2(1-r^2)$$

(S1) $$\int_0^{2\pi}\int_0^1 \frac{\frac{a^3 r^3 \cos^3\theta\, c^2}{a^2} + \frac{b^3 r^3 \sin^3\theta\, c^2}{b^2} + \frac{z^3 \cdot c^2}{c^2}}{c\sqrt{1-r^2}}\, dr\, d\theta$$

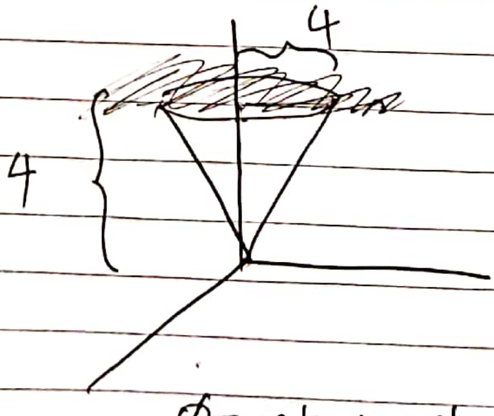
$$\int_0^{2\pi}\int_0^1 c^2(1-r^2)\, dr\, d\theta = 0.$$

(S2) : $$-c^2 \int_0^{2\pi}\int_0^1 \frac{a^3\cos^3\theta r^3 + b^3\sin^3\theta r^3}{a^2 c(\sqrt{1-r^2})} - c^2(1-r^2)\, dr\, d\theta.$$

$$= 0.$$

47. Evaluate $\iint \vec{F}\cdot\hat{n}\, ds$ over the entire surface of the region above the x-y plane bounded by the cone $z^2 = x^2+y^2$ and the plane $z=4$ if $\vec{F} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

$$\nabla\cdot\vec{F} = 4z + xz^2 + 3$$



$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$

$z^2 = \rho^2 \Rightarrow z = \rho$
$\rho = 0$ to $\rho = 4$
$z = 0$ to $z = \rho$

$\phi = \pi/4$ to $\pi/2$ ~~$\phi = 0$ to $\phi = 2\pi$~~

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$$= \iiint (4z + xz^2 + 3)\, dV$$

$$= \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{4} \int_{z=0}^{\rho} (\rho^2 \sin\phi)(4z + \rho\cos\phi z^2 + 3)\, dz\, d\rho\, d\phi$$

$$= \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{4} \int_{z=0}^{\rho} 4\rho^2 z \sin\phi + \rho^3 \sin\phi\cos\phi z^2 + 3\rho^2 \sin\phi\, dz\, d\rho\, d\phi$$

$$= \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{4} 2\rho^4 \sin\phi + \frac{\rho^6 \sin\phi\cos\phi}{3} + 3\rho^3 \sin\phi\, d\rho\, d\phi$$

$$= \int_{\phi=\pi/4}^{\pi/2} \frac{2\times 4^5}{5}\sin\phi + \frac{4^7}{7\times 3}\sin\phi\cos\phi + \frac{3\times 4^4}{4}\times\sin\phi\, d\phi$$

$$= \frac{2\times 4^5}{5}\left[-\cos\phi\right]_{\pi/4}^{\pi/2} + \frac{4^7}{21\times 2}\left[\frac{-\cos 2\phi}{2}\right]_{\pi/4}^{\pi/2} + 3\times 4^3\left[-\cos\phi\right]_{\pi/4}^{\pi/2}$$

$$= \frac{2^{11}}{5}\left(\frac{1}{\sqrt{2}}\right) + \frac{4^6}{21}\left(0+1\right) + 3\times 4^3\left(\frac{1}{\sqrt{2}}\right)$$

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